§ 1 Fundamental Theorems

1.1 Line Integrals

A natural way to construct the integral of a complex function over a curve in the complex plane is to link it to line integrals in $\mathbb{R}^2$ as already seen in vector calculus.

We may understand this in two steps:

Definition Consider a complex function $f(t) = u(t) + iv(t)$, for $t \in [a, b] \subset \mathbb{R}$, and u and v real valued functions. If f is a continuous function, we may define

$$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \tag{1}$$

Remark This definition, combined with the elementary properties of addition and multiplication in $\mathbb{C}$ we saw in Chapter 1 Lecture, means that the integral has many intuitive properties that are reminiscent of the properties of integrals of real functions. Let us mention a few without proof, as these proofs are elementary:

– Let $c \in [a, b]$, $\lambda \in \mathbb{C}$ and let f be continuous on $[a, b]$. Then

$$\int_a^c f(t)\,dt + \int_c^b f(t)\,dt = \int_a^b f(t)\,dt$$

$$\int_a^b \lambda\, f(t)\,dt = \lambda \int_a^b f(t)\,dt$$

$$Re\left(\int_a^b f(t)\,dt\right) = \int_a^b Re\,(f(t))\;dt \quad , \quad Im\left(\int_a^b f(t)\,dt\right) = \int_a^b Im\,(f(t))\;dt$$

– Although the following property is also intuitive, let us prove that:

$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\;dt \tag{2}$$

If $\int_a^b f(t)\,dt = 0$, the inequality is trivial.

For $\int_a^b f(t)\,dt \neq 0$, let $\theta = \arg\left(\int_a^b f(t)\,dt\right)$, then $\int_a^b f(t)\,dt = \left|\int_a^b f(t)\,dt\right| e^{i\theta}$ and

$$\left|\int_a^b f(t)\,dt\right| = Re\left(e^{-i\theta}\int_a^b f(t)\,dt\right) = Re\left(\int_a^b e^{-i\theta} f(t)\,dt\right) = \int_a^b Re\left(e^{-i\theta} f(t)\right)\,dt \leq \int_a^b |f(t)|\;dt \quad \square$$

With this preliminary step in place, we are ready to define integration on a general curve in $\mathbb{C}$.

Definition Let γ be a piecewise differentiable arc in the complex plane, with parametric equation

$$\gamma\ :\ z = z(t)\ ,\quad a < t < b$$

If the function f is continuous on γ, then $f(z(t))$ is continuous on (a, b), and we define the *integral of f on γ* as the line integral

$$\int_\gamma f(z)\,dz := \int_a^b f(z(t))\,\frac{dz}{dt}\,dt \tag{3}$$

where the integral $\int_a^b$ may have to be split to match the intervals in which z is differentiable.

Remark The definition above only makes sense if the integral is independent of the way the arc γ is parameterized. This is simple to check, using the rules for the change of variables for integrals of real valued functions. Imagine that another parameterization for γ is given by

$$\gamma \;:\; \tau \in (\alpha,\, \beta) \mapsto z\left(t(\tau)\right)$$

with $t \;:\; \tau \in (\alpha,\, \beta) \mapsto t(\tau) \in (a,\, b)$ piecewise differentiable. Then,

$$\begin{aligned}\int_\gamma f(z)\, dz = \int_a^b f\left(z(t)\right)\, z'(t)\, dt &= \int_\alpha^\beta f\left(z\left(t(\tau)\right)\right) \frac{dz}{dt}\frac{dt}{d\tau}\, d\tau \\ &= \int_\alpha^\beta f\left(z\left(t(\tau)\right)\right) \frac{dz}{d\tau}\left(t(\tau)\right)\, d\tau \quad \square\end{aligned}$$

1.3 Line Integrals as Functions of Arcs

Definition Let $\gamma \,:\, z = z(t),\, t \in (a,\, b)$. We define the *opposite arc*, written $-\gamma$, by

$$-\gamma \,:\, z = z(-t),\; t \in (-b,\, -a)$$

Then,

$$\int_{-\gamma} f(z)\, dz = \int_{-b}^{-a} f\left(z(-t)\right) \frac{d}{dt}\left[z(-t)\right]\, dt = -\int_{-b}^{-a} f\left(z(-t)\right)\left(z'\left(-t\right)\right)\, dt = \int_b^a f\left(z(t)\right) z'(t)\, dt$$

where the last equality is obtained with a simple change of variable. Hence

$$\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz \tag{4}$$

Elementary Properties :

- Linearity as an operator on functions

Let f and g be two continuous functions on the piecewise differentiable arc γ, and $(\alpha,\, \beta) \in \mathbb{C}^2$

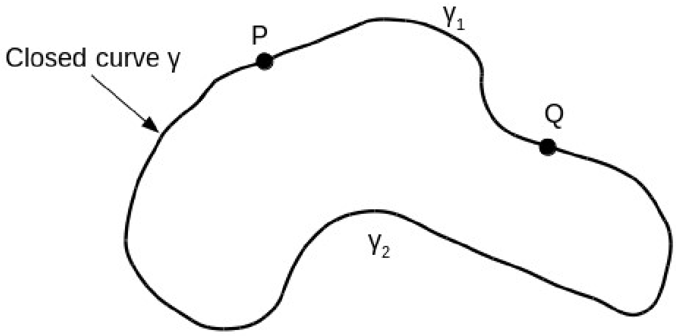
$$\int_\gamma \left(\alpha f + \beta g\right)\, dz = \alpha \int_\gamma f\, dz + \beta \int_\gamma g\, dz \tag{5}$$

- Linearity as an operator on curves

Consider an arc γ which can be subdivided into two piecewise-differentiable arcs γ_1 and γ_2, and f a continuous function on γ. Then

$$\int_\gamma f\, dz = \int_{\gamma_1} f\, dz + \int_{\gamma_2} f\, dz = \int_{\gamma_2} f\, dz + \int_{\gamma_2} f\, dz \tag{6}$$

We can use this property to show that an integral over a closed curve does not depend on the starting point on the curve. Indeed, consider two such points P and Q, corresponding to different parameterizations, as shown in the figure. If we call γ_1 the part of γ from P to Q, and γ_2 the part of γ from Q to P,



The expression in the middle corresponds to the evaluation of the integral starting from the point P, while the expression on the right corresponds to the evaluation of the integral starting from the point Q.

We conclude this section with a very simple example which will play a fundamental role in the rest of this course.

Example Let $a \in \mathbb{C}$, and consider the integral

$$\int_\gamma \frac{dz}{z-a}$$

where γ is the closed circle with radius R and centered in a. A simple parameterization for γ is $\gamma\,:\,\theta \in [0,\,2\pi) \mapsto z(\theta) = Re^{i\theta} + a$. Thus

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} i\,d\theta = 2\pi i$$

- Line integrals with respect to x and y

The line *integral with respect to* $\overline{z}$ is defined as

$$\int_\gamma f(z)\,d\overline{z} := \overline{\int_\gamma \overline{f(z)}\,dz} \tag{7}$$

Line integrals with respect to $x = Re\,(z)$ and $y = Im\,(z)$ along the arc γ are then naturally constructed as

$$\int_\gamma f(z)\,dx = \frac{1}{2}\left(\int_\gamma f(z)\,dz + \int_\gamma f(z)\,d\overline{z}\right)\,,\;\int_\gamma f(z)\,dy = \frac{1}{2i}\left(\int_\gamma f(z)\,dz - \int_\gamma f(z)\,d\overline{z}\right) \tag{8}$$

If we then write $f(z) = u(x,\,y) + iv(x,\,y)$, with $z = x + iy$, we have

$$\int_\gamma f(z)\,dz = \int_\gamma f(z)\,dx + i\int_\gamma f(z)\,dy = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx) \tag{9}$$

which can be viewed as another definition for $\int_\gamma f(z)\,dz$, involving only line integrals of scalar functions, as already introduced in vector calculus. Note that we have just reduced the complex integral $\int_\gamma f(z)\,dz$ to line integrals of the form $\int_\gamma P(x,\,y)\,dx + Q(x,\,y)\,dy$ as in Eq (9).

- Independence of Path

Recall that the integral $\int_\gamma f(z)\,dz$ is said to be *path independent* if it has the same value for any two paths with the same endpoints.

Theorem 1. (Independence of Path) Let Ω be an open connected set of $\mathbb{R}^2$, P and Q be complex-valued continuous functions defined on Ω, and γ be a piecewise differentiable arc in Ω. Then the integral $\int_\gamma P\,dx + Q\,dy$ depends only on the end points of γ iff there exists a function $U(x, y)$ on Ω with the partial derivatives $P(x, y) = \frac{\partial U}{\partial x}$, $Q(x, y) = \frac{\partial U}{\partial y}$.

Proof ($\Leftarrow$) If such a U exists, then for any arc γ between the points $(x(a), y(a))$ and $(x(b), y(b))$,

$$\int_\gamma P\,dx+Q\,dy = \int_a^b \left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}\right) dt = \int_a^b \frac{d}{dt}\left[U(x(t), y(t))\right] dt = U(x(b), y(b)) - U(x(a), y(a))$$

depends only on the end points of γ.

($\Rightarrow$) Conversely, if $\int_\gamma P(x, y)\,dx + Q(x, y)\,dy$ only depends on the end points of γ, we can construct a single valued function U by fixing a point $(x_0, y_0) \in \Omega$, and defining for each $(x_1, y_1) \in \Omega$ that

$$U(x_1, y_1) = \int_\gamma P(x, y)\,dx + Q(x, y)\,dy$$

where γ is any arc between (x_0, y_0) and (x_1, y_1). We now show that U satisfies the conditions of the theorem.

Consider the point $(x_1 + \Delta x, y_1)$, and any arc γ' between (x_0, y_0) and $(x_1 + \Delta x, y_1)$. For Δx sufficiently small, there exists an arc $\gamma''(t) = (x_1, y_1) + (1-t)(\Delta x, 0)$ in Ω between (x_1, y_1) and $(x_1 + \Delta x, y_1)$ for $t \in [0, 1]$. Since the integral is path independent, we can write

$$\begin{aligned}
U(x_1 + \Delta x, y_1) &= \int_{\gamma'} P(x, y)\,dx + Q(x, y)\,dy \\
&= \int_\gamma P(x, y)\,dx + Q(x, y)\,dy + \int_{\gamma''} P(x, y)\,dx \\
&= U(x_1, y_1) + \int_{\gamma''} P(x, y)\,dx
\end{aligned}$$

Constructing arcs γ'' in this manner for all small Δx, we may write

$$\lim_{\Delta x\to 0} \frac{U(x_1 + \Delta x, y_1) - U(x_1, y_1)}{\Delta x} = \lim_{\Delta x\to 0} \frac{1}{\Delta x} \int_{x_1}^{x_1+\Delta x} P(x, y_1)\,dx = P(x_1, y_1)$$

where the last equation follows from the continuity of P. We thus have $\frac{\partial U}{\partial x}(x_1, y_1) = P(x_1, y_1)$ for each $(x_1, y_1) \in \Omega$. With a very similar proof, we would show that $\frac{\partial U}{\partial y}(x_1, y_1) = Q(x_1, y_1)$ for each $(x_1, y_1) \in \Omega$. which concludes our proof. $\square$

Remark Let $f(z) = u(x, y)+iv(x, y)$, $P(x, y) = u(x, y)+iv(x, y)$, and $Q(x, y) = i\left(u(x, y) + iv(x, y)\right)$. Since

$$\int_\gamma f(z)\,dz = \int_\gamma f(z)\,(dx + idy) = \int_\gamma P(x, y)\,dx + Q(x, y)\,dy$$

and by the Theorem 1, the integral on the right-hand side depends on the end points if and only if there exists $F(x,y)$ such that $P(x,y) = \dfrac{\partial F}{\partial x}$, and $iP(x,y) = Q(x,y) = \dfrac{\partial F}{\partial y}$. If such an F exists, then

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

Writing $F(z) = U(x,y) + iV(x,y)$, the equality above becomes Cauchy-Riemann equations for U and V. So F is analytic, with derivative f. We have proven the following theorem.

Theorem (Fundamental Theorem of Calculus for Integrals in $\mathbb{C}$) Let Ω be an open connected subset of $\mathbb{C}$, f be a continuous function defined on Ω, and γ be a piecewise differentiable arc in Ω. Then the integral $\displaystyle\int_\gamma f(z)\,dz$ depends only on the end points of γ iff there exists an analytic function F, called a *primitive* of f, such that $F'(z) = f(z)$ for all $z \in \Omega$.

Proof If F is an analytic function such that $f(z) = F'(z)$ for all $z \in \Omega$, then, since

$$\oint_\Gamma f(z)\,dz = \oint_\Gamma F'(z)\,dz = 0 \quad \text{for each closed curve } \Gamma \subset \Omega, \tag{10}$$

the integral $\displaystyle\int_\gamma f(z)\,dz$ is path independent.

Conversely, if f is a continuous function on an open connected set Ω and is such that

$$\oint_\Gamma f(z)\,dz = 0$$

for each closed contour Γ in Ω, then f has a primitive by using a similar argument as in the proof of Theorem 1.

Example Let $n = 0,\ 1,\ 2,\ \ldots,$ and $a \in \mathbb{C}$. Since

$$(z-a)^n = \frac{d}{dz}\left[\frac{(z-a)^{n+1}}{n+1}\right] \quad \text{and} \quad \frac{(z-a)^{n+1}}{n+1} \quad \text{is entire,}$$

$\displaystyle\int_\gamma (z-a)^n\,dz = 0$ for any closed curve γ in $\mathbb{C}$.

– When $n = -1$ and γ is a closed curve around a, $\displaystyle\int_\gamma (z-a)^n\,dz = 2k\pi i \neq 0$ for some $k \in \mathbb{Z}\backslash\{0\}$.

– When $n = -2,\ -3, \ldots,$ $\displaystyle\int_\gamma (z-a)^n\,dz = 0$ for any closed curve γ in $\mathbb{C} \setminus \{a\}$.

- Integration with respect to arc length

We will often encounter *integrals with respect to arc length*, defined by

$$\int_\gamma f(z)\,ds := \int_\gamma f(z)\,|dz| = \int_a^b f(z(t))\,|z'(t)|\,dt \tag{11}$$

As before, this only makes sense if the integral is independent of the parameterization. This

$$\int_{-\gamma} f(z)\,ds = \int_\gamma f(z)\,ds$$

The length of a curve γ in the complex plane is given by

$$L(\gamma) = \int_\gamma ds = \int_\gamma |dz|$$

Finally, using Eq. (2), we have the inequality:

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\, |dz| \leq L(\gamma) \sup_{z\in\gamma} |f(z)| \tag{12}$$

1.2 Rectifiable Arcs

Consider the arc $\gamma : z = z(t)$, $a \leq t \leq b$. We have seen a way to define its length if it is piecewise differentiable. A more general definition is given by the least upper bound of all sums of the form

$$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \cdots + |z(t_n) - z(t_{n-1})|$$

with $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$. If this least upper bound is finite, γ is said to be a *rectifiable* arc. Any piecewise differentiable arc is rectifiable, and in that case the two definitions of length are equivalent. The integral of a continuous function f on a rectifiable arc may be defined as

$$\int_\gamma f(z)\, dz = \lim_{\substack{N\to\infty, \\ |z(t_k) - z(t_{k-1})| \to 0}} \sum_{k=1}^{N} f\left(z(t_k)\right)\left(z(t_k) - z(t_{k-1})\right)$$

In this course, we will never have to consider arcs which are not piecewise differentiable, but it is important to know that many of the theorems hold with weaker assumptions on γ.

Cauchy's Theorems

1.4 Cauchy's Theorem for a Rectangle

Theorem 2. (Cauchy-Goursat Theorem) Let $R = \{z = x + iy \in \mathbb{C} \mid a \leq x \leq b,\ c \leq y \leq d\}$ be the rectangle in the complex plane and ∂R be the boundary curve of R oriented in the counterclockwise direction. If a function f is analytic on an open set U containing R, then

$$\int_{\partial R} f(z)\, dz = 0.$$

Proof For any closed rectangle $\widetilde{R} \subseteq R$, let $\eta(\widetilde{R})$ be the integral of f on $\partial\widetilde{R}$ defined by

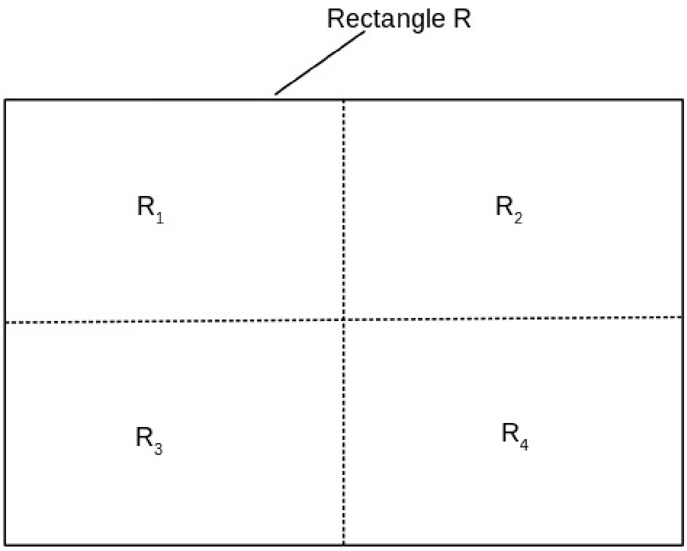
$$\eta(\widetilde{R}) = \int_{\partial\widetilde{R}} f(z)\, dz$$

Bisecting R into four congruent rectangles R_1, R_2, R_3, and R_4, as shown in the figure, and using that the integrals along shared sides cancel each other, we can write

$$\eta(R) = \eta(R_1) + \eta(R_2) + \eta(R_3) + \eta(R_4) \implies \exists\, i \text{ such that } |\eta(R)| \leq 4\,|\eta(R_i)|$$

Let $R^{(1)}$ denote this subrectangle such that

$$\frac{1}{4}\left|\eta\left(R^{(1)}\right)\right| \geq |\eta(R)|$$



For each $n = 1, 2 \ldots$, we repeat the bisection process in that $R^{(n)}$ to obtain a nested sequence of nonempty closed subrectangles $\{R^{(n)}\}_{n=1}^{\infty}$ such that

$$\left|\eta\left(R^{(n)}\right)\right| \geq \frac{1}{4}\left|\eta\left(R^{(n-1)}\right)\right| \geq \cdots \geq \frac{1}{4^n}\left|\eta\left(R\right)\right|$$

and $\displaystyle\bigcap_{n=1}^{\infty} R^{(n)} \neq \emptyset$.

Let z_0 be the unique point such that $\{z_0\} = \displaystyle\bigcap_{n=1}^{\infty} R^{(n)}$. For any $\varepsilon > 0$, since f is analytic at $z_0 \in R$, $\exists\, \delta > 0$ such that

if $|z - z_0| < \delta$, then $\left|\dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon \iff |f(z) - f(z_0) - f'(z_0)\,(z - z_0)| < \varepsilon\,|z - z_0|$

With this δ, $\exists N \in \mathbb{N}$ such that

$$\forall\, n \geq N\ ,\ R^{(n)} \subset \{z \in \mathbb{C} \mid\ |z - z_0| < \delta\} \tag{13}$$

Also since $f(z_0) + f'(z_0)\,(z - z_0)$ has an entire primitive, $\displaystyle\int_{\partial R^{(n)}} \left(f(z_0) + f'(z_0)\,(z - z_0)\right)\,dz = 0$ by the Fundamental Theorem of Calculus, we can write

$$\eta\left(R^{(n)}\right) = \int_{\partial R^{(n)}} f(z)\,dz = \int_{\partial R^{(n)}} \left(f(z) - f(z_0) - f'(z_0)\,(z - z_0)\right)\,dz$$

so that

$$\left|\eta\left(R^{(n)}\right)\right| \leq \varepsilon \int_{\partial R^{(n)}} |z - z_0|\ |dz| \leq \varepsilon\, d_n L_n$$

where d_n is the length of the diagonal of $R^{(n)}$, and L_n is the length of its perimeter $\partial R^{(n)}$. Finally, if d is the length of the diagonals of R and L the length of its perimeter ∂R, since

$$d_n = \frac{d}{2^n}\ ,\ L_n = \frac{L}{2^n}$$

we conclude that

$$|\eta(R)| \leq 4^n \left|\eta\left(R^{(n)}\right)\right| \leq \varepsilon\, dL$$

Since ε is arbitrarily small, $\eta(R) = 0$. $\square$

Our goal is to now generalize this result for cases in which f may not be analytic at a finite number of points ζ_i inside R :

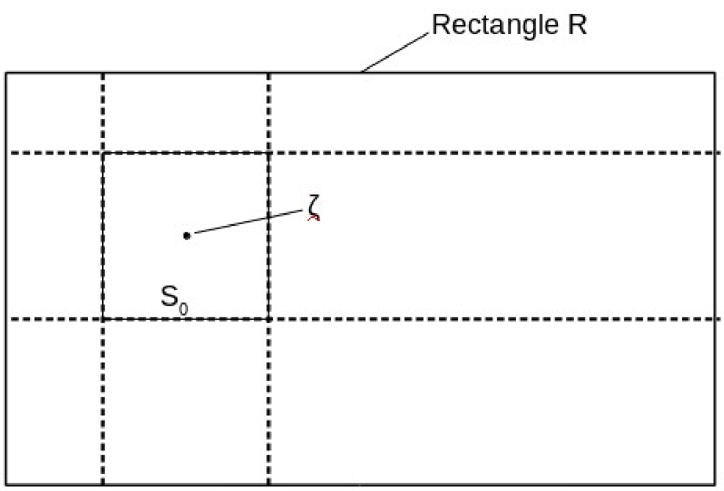
Theorem 3. Let f be an analytic function on the set R' obtained from a rectangle R by omitting a finite number of interior points ζ_i. If f satisfies the condition

$$\lim_{z\to\zeta_i} (z-\zeta_i)\, f(z) = 0$$

for all i, then

$$\int_{\partial R} f(z)\, dz = 0$$

Proof Without loss of generality, we assume that f is not analytic at only one point ζ in R. We then subdivide R as shown in the figure, where S_0 **is a square with center** ζ.



Using Theorem 2, we have

$$\int_{\partial R} f(z)\, dz = \int_{\partial S_0} f(z)\, dz$$

Now, $\forall\, \varepsilon > 0$, since $\lim\limits_{z\to\zeta} (z-\zeta)\, f(z) = 0$, we may choose S_0 small enough such that

$$\text{if } z \in \partial S_0\text{ , then } |(z-\zeta)\, f(z)| \leq \varepsilon \implies |f(z)| \leq \frac{\varepsilon}{|z-\zeta|}$$

Hence,

$$\left|\int_{\partial R} f(z)\, dz\right| = \varepsilon \int_{\partial S_0} \frac{|dz|}{|z-\zeta|} \leq \varepsilon\, \frac{4\ell}{\ell/2} = 8\varepsilon$$

where ℓ is the length of a side of the square. Since ε is arbitrarily small, $\int_{\partial R} f(z)\, dz = 0$. $\square$

1.5 Cauchy's Theorem in a Disk

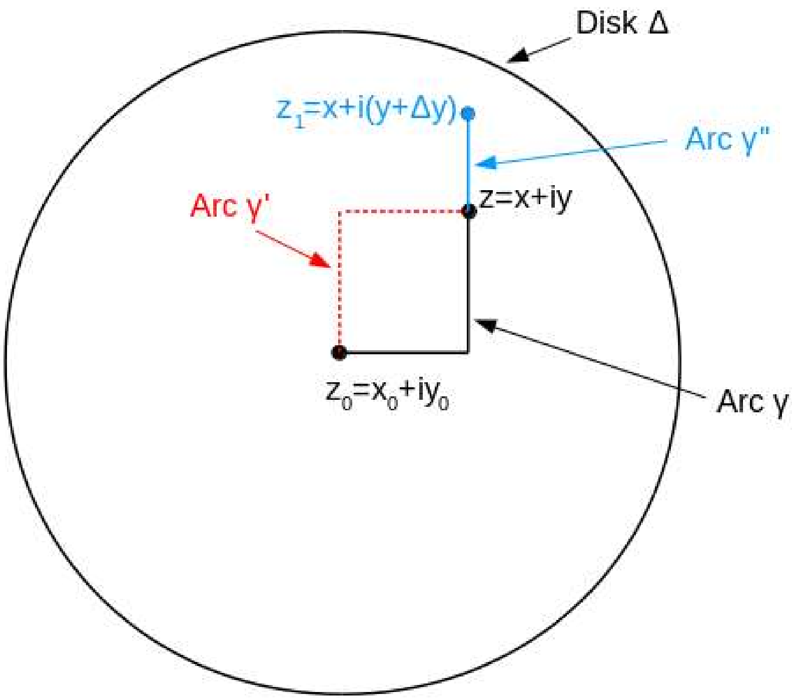
Theorem 4. If f is analytic in an open disk Δ, then

$$\int_\gamma f(z)\, dz = 0 \quad \text{for every closed curve } \gamma \text{ in } \Delta.$$

Proof Let $z_0 = x_0 + iy_0$ be the center of the disk Δ. For each $z = x + iy \in \Delta$, we define

$$F(z) = \int_\gamma f(\zeta)\, d\zeta = \int_\gamma f(\zeta)\, dx + \int_\gamma if(\zeta)\, dy$$

where γ is the arc that is horizontal from (x_0, y_0) to (x, y_0) and vertical from (x, y_0) to (x, y) in the disk Δ.



Thus we have

$$\frac{\partial F}{\partial y} = \lim_{\Delta y \to 0} \frac{F(x, y + \Delta y) - F(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{\gamma''} f(\zeta)\, d\zeta = \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{\gamma''} if(\zeta)\, dy = if(z)$$

where γ'' is the vertical line from (x, y) to $(x, y + \Delta y)$ as in the above figure.

Now, by Cauchy's theorem on rectangles, one can also write

$$F(z) = \int_{\gamma'} f(\zeta)\, d\zeta = \int_{\gamma'} f(\zeta)\, dx + \int_{\gamma'} if(\zeta)\, dy$$

where γ' is the arc that is vertical from (x_0, y_0) to (x_0, y) and horizontal from (x_0, y) to (x, y) in the disk Δ. Thus, applying the same reasoning, we can also find

$$\frac{\partial F}{\partial x} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta y} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{\widetilde{\gamma}} f(\zeta)\, d\zeta = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{\widetilde{\gamma}} f(\zeta)\, dx = f(z)$$

where $\widetilde{\gamma}$ is the horizontal line from (x, y) to $(x + \Delta x, y)$. We conclude that $\dfrac{\partial F}{\partial x} = -i\dfrac{\partial F}{\partial y}$ which implies that F is analytic and is a primitive of f in Δ, so that by the fundamental theorem of calculus

$$\int_\gamma f(z)\, dz = 0 \quad \text{for any closed curve } \gamma \text{ in } \Delta. \quad \square$$

We are now ready to prove Cauchy's theorem in its full extent, as stated in the following.

Theorem 5. (Cauchy's Theorem) Let f be an analytic function in the open connected set Δ' obtained by omitting a finite number of points ζ_i from an open disk Δ. If f satisfies the condition

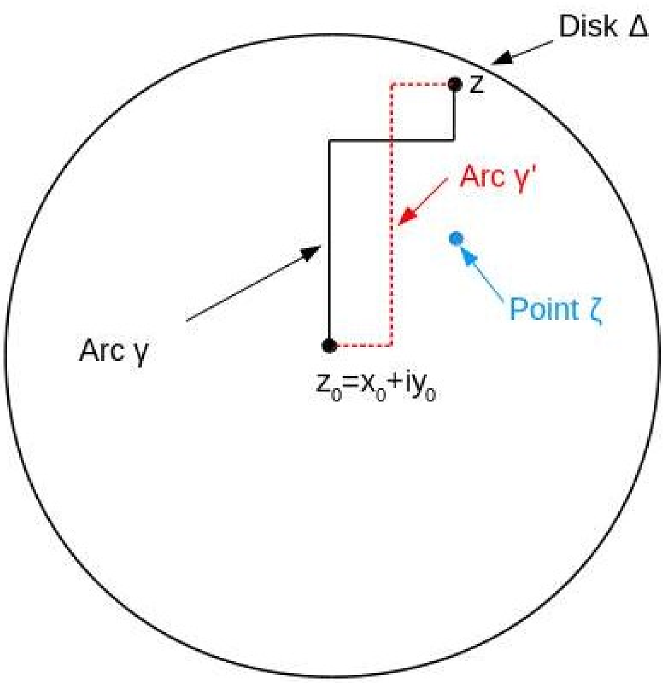
$$\lim_{z\to\zeta_i}(z-\zeta_i)\,f(z)=0$$

for all i, then

$$\int_\gamma f(z)\,dz=0 \quad \text{for any closed rectifiable arc } \gamma \text{ in } \Delta'.$$

Proof Without loss of generality, we can again assume that there is only one special point ζ in Δ. We define $F(z)=\displaystyle\int_{\gamma'} f(z)\,dz$ in a similar way as before; we just have to be careful with the location of ζ with respect to the arcs we used in the proof.

First case: ζ lies neither on the line $x=x_0$ nor on the line $y=y_0$, where $z_0=x_0+iy_0$ is the center of Δ and the initial point of γ. Then it is possible to construct a path γ from z_0 to any $z\neq\zeta$ made only of horizontal and vertical line segments (three segments may be needed) with the last segment a vertical segment and where γ does not go through ζ as shown in the figure.



It is then easy to show, in the same way as before, that $F_y(z)=if(z)$.

We know by Cauchy's theorem on a rectangle that $F(z)=\displaystyle\int_{\gamma'} f(z)\,dz$, with γ' shown in the figure, and that therefore $F_x(z)=f(z)$. We conclude that F is analytic, so $\displaystyle\int_\gamma f(z)\,dz=0$ for any closed curve in Δ'.

Second case: ζ lies on the line $x=x_0$ or on the line $y=y_0$. In that case, one just moves the starting point z_0 for the definition of F away from x_0+iy_0 to return to the first case. $\square$

§2 Cauchy Integral Formulas

2.1 The Index of a Point with Respect to a Closed Curve

Lemma 1. Let $z \in \mathbb{C}$ and γ be a piecewise differentiable closed curve that does not pass through z. Then there exists an integer $k \in \mathbb{Z}$ such that

$$\int_\gamma \frac{d\zeta}{\zeta - z} = 2\pi k i.$$

Proof Let $\gamma : \zeta = \zeta(t)$, $a \le t \le b$, and consider the function

$$f(t) = \int_a^t \frac{1}{\zeta(u) - z} \frac{d\zeta}{du}\, du$$

Since γ does not pass through z, f is defined and continuous on $[a, b]$. Furthermore, for all t such that $\dfrac{d\zeta}{dt}(t)$ is continuous, we can write

$$f'(t) = \frac{1}{\zeta(t) - z}\frac{d\zeta}{dt} \iff \frac{d}{dt}\left[e^{-f(t)}\left(\zeta(t) - z\right)\right] = 0$$

Since γ is piecewise differentiable and since $e^{-f(t)}(\zeta(t) - z)$ is continuous on γ, we have

$$e^{-f(t)}\left(\zeta(t) - z\right) = \text{constant} = e^{-f(a)}\left(\zeta(a) - z\right) = \left(\zeta(a) - z\right) \implies e^{f(t)} = \frac{\zeta(t) - z}{\zeta(a) - z}$$

Also, since γ is a closed curve, $\gamma(b) = \gamma(a)$ which implies that

$$e^{f(b)} = e^{f(a)} = 1 \iff \exists\, k \in \mathbb{Z} \text{ such that } \int_\gamma \frac{d\zeta}{\zeta - z} = f(b) = 2\pi k i \quad \square$$

Definition The *index* of the point z with respect to the closed curve γ is the number

$$n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z} \tag{14}$$

n can be viewed as a quantity measuring the number of times a closed curve winds around a fixed point not on it. For this reason, n is often called the *winding number*.

Theorem Let γ be a piecewise differentiable closed curve. The function $f : z \mapsto n(\gamma, z)$ is constant on each open connected set of $\mathbb{C} \setminus \{\gamma\}$, and zero if this set is unbounded.

Proof The function

$$f : z \mapsto \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}$$

is integer valued on any open connected subset Ω of $\mathbb{C} \setminus \{\gamma\}$, and continuous on these sets. Since the image $f(\Omega)$ of any such set Ω is also connected, and the only connected subset of the integers contains at most one point, f is constant. In addition, for $|z|$ sufficiently large, there is a disk $D_R(0)$ such that $|z| > R$ and γ is contained in $D_R(0)$. Since $1/(\zeta - z)$ is analytic for all $\zeta \in D_R(0)$, $n(\gamma, z) = 0$ by the Cauchy's theorem, and the result holds for the entire region by continuity. $\square$

2.2 Cauchy's Integral Formula

Theorem 6. Let f be analytic in an open disk Δ, γ be a closed curve in Δ, and $n(\gamma, z)$ be the index of z with respect to γ. Then

$$n(\gamma, z)\, f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for each } z \in \Delta \setminus \{\gamma\}. \tag{15}$$

Proof For each $z \in \Delta \setminus \{\gamma\}$, we consider the function $F : \Delta \setminus \{z\} \to \mathbb{C}$ defined by

$$F(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z} \quad \text{for each } \zeta \in \Delta \setminus \{z\}$$

Since $f(\zeta) - f(z)$, $\zeta - z \neq 0$ are analytic, and $\zeta - z \neq 0$ in $\Delta \setminus \{z\}$, F is analytic on $\Delta \setminus \{z\}$, and satisfies that

$$\lim_{\zeta \to z} F(\zeta)\,(\zeta - z) = 0,$$

Hence, by Cauchy's theorem (e.g. Theorem 5), we have $\displaystyle\int_\gamma F(\zeta)\,d\zeta = 0$, i.e.

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z) \int_\gamma \frac{d\zeta}{\zeta - z} = 2\pi i\, n(\gamma,\, z)\, f(z) \quad \square$$

Remark It is easy to show that the Cauchy's integral formula (Theorem 6) still holds if f is an analytic function in $\Delta \setminus \{\xi_i\}_{i=1}^n$, and satisfies that $\lim\limits_{z \to \xi_i} (z - \xi_i)\ f(z) = 0$ for each $1 \leq i \leq n$.

Note that Cauchy's formula gives an expression for $f(z)$ only knowing that f is analytic in Δ and knowing the values of f on γ. This will be useful to prove many key theorems, and to study the local properties of functions. Here is a direct illustration:

Theorem (Mean-Value Property for Analytic Functions) The value of an analytic function f at z is equal to the average of its values around any circle $|\zeta - z| = R$ inside the domain where it is analytic.

Proof The result comes directly from Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi} \int_0^{2\pi} f\left(z + Re^{i\theta}\right)\,d\theta \quad \square$$

Remark You probably came across a similar theorem for harmonic functions of real variables. The connection is clear, through the Cauchy-Riemann equations.

2.3 Higher Derivatives

It is tempting to differentiate Cauchy's formula under the integral sign to obtain analogous formulae for the derivatives of f. To do so, we need a short lemma regarding that operation:

Lemma 3. Consider an open connected set Ω of $\mathbb{C}$, and γ an arc in Ω. If φ is continunous on γ, then

$$F_n(z) = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n}\,d\zeta \quad \text{is analytic in } \Omega \setminus \{\gamma\}$$

and its derivative is $F_n'(z) = nF_{n+1}(z)$ for each $n = 0,\, 1,\, 2, \ldots$.

Proof We prove this lemma by induction.

- The lemma is true for $n = 0$.
- Let us assume that it holds for $n - 1$: F_{n-1} is analytic on $\Omega \setminus \{\gamma\}$ for any φ continuous on γ, and $F_{n-1}'(z) = (n-1)\, F_n(z)\ \forall\, z \in \Omega \setminus \{\gamma\}$
- Let $z_0 \in \Omega \setminus \{\gamma\}$, and consider a neighborhood $D_\delta(z_0)$ that does not meet γ, and inside that neighborhood a smaller neighborhood $D_{\delta/2}(z_0)$. Observe that

$$z \in D_{\delta/2}(z_0) \implies \begin{cases} |z - z_0| < \delta/2 \\ |\zeta - z\,| > \delta/2\ ,\ \forall\, \zeta \in \gamma \end{cases}$$

For any continuous function φ continuous on γ, we may write

$$F_n(z) - F_n(z_0) = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n}\, d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n}\, d\zeta = \int_\gamma \frac{\varphi(\zeta)\,(\zeta - z + z - z_0)}{(\zeta - z)^n\,(\zeta - z_0)}\, d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n}\, d\zeta$$
$$= \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}\,(\zeta - z_0)}\, d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n}\, d\zeta + (z - z_0) \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n\,(\zeta - z_0)}\, d\zeta$$

Let $\psi(\zeta) := \dfrac{\varphi(\zeta)}{\zeta - z_0}$, and rewrite the above equality as

$$F_n(z) - F_n(z_0) = \left[\int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^{n-1}}\, d\zeta - \int_\gamma \frac{\psi(\zeta)}{(\zeta - z_0)^{n-1}}\, d\zeta\right] + (z - z_0) \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n}\, d\zeta \tag{16}$$

Now, $\forall\, z \in D_{\delta/2}(z_0)$,

$$\left|(z - z_0) \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n}\, d\zeta\right| \le |z - z_0| \left(\frac{2}{\delta}\right)^n \int_\gamma |\psi(\zeta)|\, |d\zeta|$$

and since ψ is continuous on γ and γ is rectifiable, so we have

$$\lim_{z \to z_0} (z - z_0) \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n}\, d\zeta = 0$$

Furthermore, we know by the induction hypothesis that the term in brackets in Eq.(16) goes to zero as $z \to z_0$. Hence, for any φ continuous on γ, F_n is continuous in z_0.

Defining

$$G_n(z) = \int_\gamma \frac{\psi(\zeta)}{(\zeta - z)^n}\, d\zeta \overset{\text{Eq (16)}}{\Longrightarrow} \frac{F_n(z) - F_n(z_0)}{z - z_0} = \frac{G_{n-1}(z) - G_{n-1}(z_0)}{z - z_0} + G_n(z)$$

By the induction hypothesis, the first term on the right goes to $G'_{n-1}(z_0) = (n-1)\, G_n(z_0)$ as $z \to z_0$, and from our previous point we also know that G_n is continuous, so we find

$$\lim_{z \to z_0} \frac{F_n(z) - F_n(z_0)}{z - z_0} = (n-1)\, G_n(z_0) + G_n(z_0) = nG_n(z_0) = nF_{n+1}(z_0) \quad \square$$

Remark The lemma gives us the following important result: Let Ω be an open connected set in $\mathbb{C}$, z_0 be a point in Ω, $D_\delta(z_0)$ be an open disk in Ω, and C be a circle with center z_0 inside $D_\delta(z_0)$. If f is analytic in Ω, then, by the Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for each } z \text{ in the interior of } C,$$

and, by applying the lemma, its derivative

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \tag{17}$$

is analytic in the interior of C. More generally,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \quad \text{for each } n = 0,\, 1,\, 2, \ldots \tag{18}$$

is analytic in the interior of C. We have therefore proven the following central result of complex analysis:

An analytic function on the open connected set Ω has derivatives of all orders in Ω, which are themselves analytic.

Some Consequences of Cauchy Integral Formulas

Morera's Theorem Let f be continuous in an open connected set Ω. If $\displaystyle\int_\gamma f(z)\,dz = 0$ for all closed curves γ in Ω, then f is analytic in Ω.

Proof Given the hypotheses of the theorem, f has a primitive in Ω. By the result we just found, f, the derivative of an analytic function in Ω, is analytic itself. $\square$

Cauchy's Estimate Suppose f is analytic in a closed disk $|z - z_0| \le R$, and bounded on the circle $\gamma : |\zeta - z_0| = R$, i.e. $\exists\, M \ge 0$ such that $|f(\zeta)| \le M$ for all $\zeta \in \gamma$. Then for each $n = 0, 1, 2, \ldots,$

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\int_\gamma \left|\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\right| |d\zeta| \le \frac{n!}{2\pi}\,\frac{M}{R^{n+1}}\,2\pi R \implies \left|f^{(n)}(z_0)\right| \le \frac{n!\,M}{R^n} \tag{19}$$

Remark This inequality is known as Cauchy's estimate. It can be used for the well-known Liouville theorem below.

Liouville's Theorem A bounded entire function is constant.

Proof Let f be an entire function bounded by M. Then, using Cauchy's estimate, we have that

$$\forall\, z \in \mathbb{C},\ \forall\, R > 0\ ,\ |f'(z)| \le \frac{M}{R} \implies |f'(z)| \le \lim_{R\to\infty}\frac{M}{R} = 0 \quad \forall\, z \in \mathbb{C}$$

Hence $f'(z) = 0$ for all $z \in \mathbb{C}$, which means that f is constant. $\square$

The Fundamental Theorem of Algebra Every polynomial of degree $n \ge 1$ has n roots.

Proof Assume that $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ does not have a root. Then $g(z) := \dfrac{1}{P(z)}$ is an entire function. Furthermore, g is bounded since

$$\lim_{|z|\to\infty}\frac{|P(z)|}{|z|^n} = |a_n| \implies \lim_{|z|\to\infty}\frac{1}{P(z)} = 0$$

By Liouville's theorem, $\dfrac{1}{P(z)}$ must be a constant equal to zero, which is not possible. Hence, P has at least one root α, and we can write

$$P(z) = (z - \alpha)\ Q(z)$$

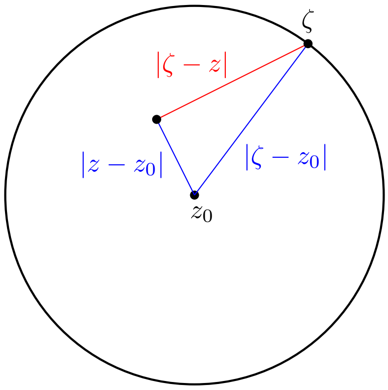
Repeating the steps for Q, we find that P must eventually have n roots. $\square$

Power Series Representation If f is analytic in an open connected set Ω which contains a closed disk $D_R(z_0)$, then f has a power series expansion at z_0,

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z - z_0)^n \quad \text{which is convergent for all } z \in D_R(z_0),\ \text{with } c_n = \frac{f^{(n)}(z_0)}{n!}$$

Proof $\forall\, z \in D_R(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < R\}$, $\forall\, \zeta \in C_R(z_0) = \{\zeta \in \mathbb{C} \mid |\zeta - z_0| = R\}$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\,\frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n = \sum_{n=0}^{\infty}(\zeta - z_0)^{-n-1}\,(z - z_0)^n$$



Since convergence is uniform in $\zeta \in C_R(z_0)$, we can use Cauchy's formula to write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C_R(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \oint_{C_R(z_0)} f(\zeta) \sum_{n=0}^{\infty} (\zeta - z_0)^{-n-1} (z - z_0)^n \, d\zeta \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{C_R(z_0)} f(\zeta)\, (\zeta - z_0)^{-n-1} \, d\zeta \right) (z - z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \square \end{aligned}$$

§ 3 Local Properties of Analytic Functions

3.1 Removable Singularities and Taylor's Theorem

We have said that Cauchy's integral formula applied to an analytic function f in $\Delta \setminus \{\xi_i\}_{i=1}^n$ as long as $\lim\limits_{z \to \xi_i} (z - \xi_i)\, f(z) = 0$ for each $1 \leq i \leq n$. We will now see that Cauchy's integral formula provides a natural way to extend such f to an analytic function $\widetilde{f}$ on the entire set Δ. In other words, $\{\xi_i\}_{i=1}^n$ are *removable singularities of f*.

Theorem 7. Let Ω be an open connected subset of $\mathbb{C}$, ξ be a point in Ω, and f be analytic in $\Omega' = \Omega \setminus \{\xi\}$. There exists an analytic function $\widetilde{f}$ in Ω which coincides with f in Ω' iff $\lim\limits_{z \to \xi} (z - \xi)\, f(z) = 0$. The extended function $\widetilde{f}$ is uniquely determined.

Proof If the extended function $\widetilde{f}$ exists, it is unique since $\widetilde{f}$ is continuous at ξ.
($\Longrightarrow$) Since $\widetilde{f}$ is continuous at ξ, we have $\lim\limits_{z \to \xi} (z - \xi)\, f(z) = \lim\limits_{z \to \xi} (z - \xi)\, \widetilde{f}(z) = 0$.

($\Longleftarrow$) Let $\Delta = D_r(\xi)$ be an open disk in Ω, let C be a circle with center ξ inside Δ, and let $F : \Delta \setminus \{\xi\} \to \mathbb{C}$ be a function defined by

$$F(\zeta) := \frac{f(\zeta) - f(z)}{\zeta - z} \quad \text{for each } \zeta \in \Delta \setminus \{\xi\}.$$

Since F has two singularities in Δ : $\zeta = z$ and $\zeta = \xi$, and since

$$\lim_{\zeta \to z} (\zeta - z)\, F(\zeta) = \lim_{\zeta \to z} (\zeta - z) \cdot \frac{f(\zeta) - f(z)}{\zeta - z} = \lim_{\zeta \to z} (f(\zeta) - f(z)) = 0$$

by continuity of f in z; and

$$\lim_{\zeta \to \xi} (\zeta - \xi)\, F(\zeta) = \lim_{\zeta \to \xi} (\zeta - \xi) \cdot \frac{f(\zeta) - f(z)}{\zeta - z} = \frac{\lim\limits_{\zeta \to \xi} (\zeta - \xi)\, f(\zeta)}{\xi - z} = 0$$

by the hypothesis of the theorem, we have

$$\int_C F(\zeta)\, d\zeta = 0 \quad \text{by Cauchy's theorem} \implies \int_C \frac{f(\zeta)-f(z)}{\zeta - z}\, d\zeta = 0$$

Hence, for any $z \neq \xi$ in Δ, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for all } z \in \Delta \setminus \{\xi\} \tag{20}$$

Now, we know from Cauchy's integral formula the right-hand side of (20) is an analytic function of z throughout the inside of C. It is therefore continuous in ξ, with value

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - \xi}\, d\zeta$$

In other words,

$$\widetilde{f}(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad , \quad \forall\, z \in \Omega \tag{21}$$

is the desired analytic extension of f in the whole open connected set Ω. $\square$

Remark Note that if f is analytic in an open connected set Ω and ξ is a point in Ω, then the function $F : \Omega \setminus \{\xi\} \to \mathbb{C}$ defined by

$$F(z) := \frac{f(z) - f(\xi)}{z - \xi} \quad \text{for } z \in \Omega \setminus \{\xi\}$$

is analytic in $\Omega \setminus \{\xi\}$. Since

$$\lim_{z \to \xi} (z - \xi)\ F(z) = 0 \quad , \quad \lim_{z \to \xi} F(z) = f'(\xi)$$

and by Theorem 7, there exists an (extended) analytic function f_1 on Ω such that

$$f_1(z) = \begin{cases} F(z) = \dfrac{f(z) - f(\xi)}{z - \xi} & \text{if } z \neq \xi \\ f'(\xi) & \text{if } z = \xi \end{cases}$$

Thus, we may thus write

$$f(z) = f(\xi) + (z - \xi)\, f_1(z) \quad \text{for all } z \in \Omega$$

This expansion for f can also be applied to f_1 : there exists an (extended) analytic function f_2 on Ω such that

$$f_2(z) = \begin{cases} \dfrac{f_1(z) - f_1(\xi)}{z - \xi} & \text{if } z \neq \xi \\ f_1'(\xi) & \text{if } z = \xi \end{cases}$$

and we may write

$$f_1(z) = f_1(\xi) + (z - \xi)\, f_2(z) \implies f(z) = f(\xi) + (z - \xi)\, f'(\xi) + (z - \xi)^2\, f_2(z) \quad \text{for all } z \in \Omega$$

Continuing the recursion, we can write the general form

$$f_{n-1}(z) = f_{n-1}(\xi) + (z - \xi)\, f_n(z) \quad \text{for all } z \in \Omega,\ n \geq 2$$

In this process, we obtained the following expansion for f :

$$f(z)=\sum_{k=0}^{n-1}\frac{f^{(k)}(\xi)}{k!}(z-\xi)^k+(z-\xi)^n f_n(z) \quad \text{for all } z\in\Omega$$

Furthermore, by direct differentiation at $z=\xi$, we have

$$f^{(n)}(\xi)=n!f_n(\xi)$$

We have just proved Taylor's theorem, stated below:

Theorem 8. (Taylor's Theorem) If f is analytic in an open connected set Ω containing ξ, then it is possible to write

$$f(z)=\sum_{k=0}^{n-1}\frac{f^{(k)}(\xi)}{k!}(z-\xi)^k+(z-\xi)^n f_n(z), \quad \text{where } f_n \text{ is analytic in } \Omega. \tag{22}$$

Remark Note that Taylor's formula, given by Eq.(22), is not a Taylor series, it is very useful nonetheless, especially because there is a simple expression for f_n in terms of f :

$$f_n(z)=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-\xi)^n(\zeta-z)}\,d\zeta \quad \text{where } C=\{z \mid |z-\xi|=r\}\subset\Omega \tag{23}$$

Proof of (23): By Cauchy's integral formula, we have

$$f_n(z)=\frac{1}{2\pi i}\int_C\frac{f_n(\zeta)}{(\zeta-z)}\,d\zeta=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-\xi)^n(\zeta-z)}\,d\zeta-\sum_{k=0}^{n-1}\frac{1}{2\pi i}\int_C\frac{f^{(k)}(\xi)}{k!\,(\zeta-\xi)^{n-k}(\zeta-z)}\,d\zeta$$

and note that for $1\le k\le n-1$, each term in the sum has the following form, up to a constant factor:

$$g_k(\xi)=\frac{1}{2\pi i}\int_C\frac{d\zeta}{(\zeta-\xi)^k(\zeta-z)}=\frac{1}{2\pi i}\int_C\frac{\varphi(\zeta)}{(\zeta-\xi)^k}\,d\zeta \quad 1\le k\le n-1$$

where $\varphi(\zeta)=1/(\zeta-\xi)$ is continuous on C. Hence, by the Lemma 4.2.3, we have

$$g_k'(\xi)=kg_{k+1}(\xi) \quad \text{for all } \xi\in\Omega\setminus C \text{ and } 1\le k\le n-1$$

Now, for each $\xi\neq z$ inside C, since

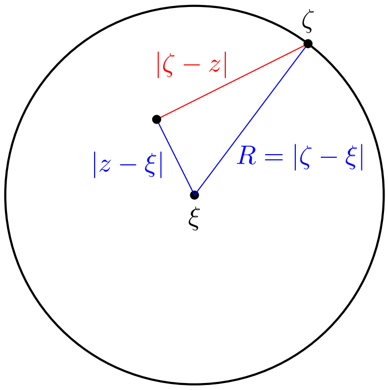
$$g_1(\xi)=\int_C\frac{d\zeta}{(\zeta-\xi)(\zeta-z)}=\frac{1}{z-\xi}\int_C\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\xi}\right)d\zeta=\frac{1}{z-\xi}(2\pi i-2\pi i)=0$$

we conclude $g_1(\xi)=0$, thus $g_1'(\xi)=0$ and $g_k(\xi)=0$ for all $2\le k\le n-1$, from which the formula (23) for f_n follows. $\square$

Corollary If f is analytic in the open connected set Ω and if there exist $\xi\in\Omega$ such that $f^{(n)}(\xi)=0$ for all $n=0,1,2,\ldots$, then $f\equiv 0$ in Ω.

Remark Since f is analytic in Ω, there is a disk $D_R(\xi)\subset\Omega$ such that $f(z)=\sum_{n=0}^{\infty}c_n(z-\xi)^n$ with $c_n=f^{(n)}(\xi)/n!=0$ for all $n=0,1,2,\ldots$, so $f\equiv 0$ in $D_R(\xi)$. Here we propose a slightly different proof as a way to practice with Taylor's formula and upper bounds.

Proof Let us first prove that $f\equiv 0$ in a disk $D_R(\xi)\subset\Omega$ with boundary $C=\{\zeta \mid |\zeta-\xi|=R\}$.



If $f^{(n)}(\xi) = 0$ for all $n = 0, 1, 2, \dots$, then by Taylor's formula we have

$$f(z) = (z-\xi)^n f_n(z) \quad \text{with} \quad f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-\xi)^n(\zeta - z)}\, d\zeta$$

Let $M = \max\{|f(z)| \mid z \in C\}$. For each $z \in D_R(\xi)$, since $|\zeta-\xi| - |z-\xi| \leq |\zeta - z|$, we have

$$|f_n(z)| \leq \frac{1}{2\pi}\frac{M}{R^n}\frac{2\pi R}{R - |z-\xi|} \implies |f(z)| \leq \left|\frac{z-\xi}{R}\right|^n \frac{MR}{R-|z-\xi|} \quad \forall\, n = 0, 1, 2, \dots \implies f(z) = 0.$$

To complete the proof, we now have to extend the result from $D_R(\xi)$ to Ω. For this purpose, consider the following two sets:

$$E_1 := \{z \in \Omega \mid f^{(n)}(z) = 0 \;\forall\, n = 0, 1, 2, \dots\}$$
$$E_2 := \{z \in \Omega \mid \exists\, n = 0, 1, 2, \dots \text{ such that } f^{(n)}(z) \neq 0\}$$

E_1 and E_2 are such that $E_1 \cap E_2 = \emptyset$. The first part of the proof shows that E_1 is an open set. Furthermore, by continuity of f and all its derivatives, E_2 is open as well. Now, since Ω is connected and $\Omega = E_1 \cup E_2$, either $E_1 = \emptyset$ or $E_2 = \emptyset$. According to the hypotheses of the theorem, $E_1 \neq \emptyset$. Therefore $E_2 = \emptyset$, and $f \equiv 0$, as desired. $\square$

3.2 Zeros and Poles

Let f be an analytic function in Ω which is not identically zero. A point $\xi \in \Omega$ is called a *zero of order $N \geq 1$* of f if, by Taylor's formula, we can write

$$f(z) = (z-\xi)^N f_N(z)$$

where f_N is analytic and $f_N(\xi) \neq 0$ which implies that $\exists\, \delta > 0$ such that $\forall\, z \in D_\delta(\xi) \setminus \{\xi\}$, $f(z) \neq 0$, i.e. the zeros of such f are isolated. This can be reformulated as the following.

Theorem Suppose $f(z) = \sum_{n=0}^{\infty} c_n (z-\xi)^n$ converges for all $z \in D_R(\xi)$. If there exists a sequence of points $\{z_k\}_{k=1}^{\infty} \subset D_R(\xi) \setminus \{\xi\}$ such that $\lim_{k\to\infty} z_k = \xi$, and $f(z_k) = 0$ for all $k \in \mathbb{N}$, then $f(z) = 0$ for all $z \in D_R(\xi)$.

Theorem (Identity Theorem) If f and g are analytic in Ω, and if $f = g$ on a set which has an accumulation point in Ω, then $\forall\, z \in \Omega$, $f(z) = g(z)$.

The theorem is immediate by looking at Taylor's formula for $f - g$, as long as we remember what an accumulation point is:

– a point z of a subset S is called an *isolated point* of S if there exists a neighborhood of z whose intersection with S reduces to the point z

– an *accumulation point* is a point of $\overline{S}$ which is not an isolated point.

Corollary If f is analytic in Ω and identically zero in a nonempty connected open subset of Ω, then $f \equiv 0$ in Ω.

Corollary If f is identically zero on an arc in Ω which does not reduce to a point, $f \equiv 0$ in Ω.

Definition f is said to have an *isolated singularity* at ξ if f is analytic in a deleted neighborhood $D_\delta(\xi) \setminus \{\xi\}$ but is *not* analytic at ξ.

Definition Suppose f has an isolated singularity at ξ.

(i) If there exists a function $\widetilde{f}$, analytic at ξ and such that $f(z) = \widetilde{f}(z)$ for all z in some deleted neighborhood $D_\delta(\xi) \setminus \{\xi\}$, we say f has a *removable singularity* at ξ. (i.e. if the value of f is "corrected" at the point ξ, it becomes analytic there).

(ii) If, for $z \neq \xi$, f can be written in the form $f(z) = A(z)/B(z)$ where A and B are analytic at ξ, $A(\xi) \neq 0$, and $B(\xi) = 0$, we say f has a *pole* at ξ. (If B has a zero of order N at ξ, we say that f has a *pole of order N*.)

(iii) If f has neither a removable singularity nor a pole at ξ, we say f has an *essential singularity* at ξ.

Remarks

(a) If f has a pole at ξ, then, by continuity, there exists $\delta > 0$ such that $f(z) \neq 0$ for all $z \in D_\delta(\xi) \setminus \{\xi\}$. Thus, $g(z) := 1/f(z)$ is analytic on $D_\delta(\xi) \setminus \{\xi\}$, and can be extended analytically on $D_\delta(\xi)$ with $g(\xi) := 0$ since $\lim\limits_{z\to\xi} (z-\xi)\, g(z) = 0$.

(b) If f has a pole of order N at ξ, then, by Taylor's formula, we can write

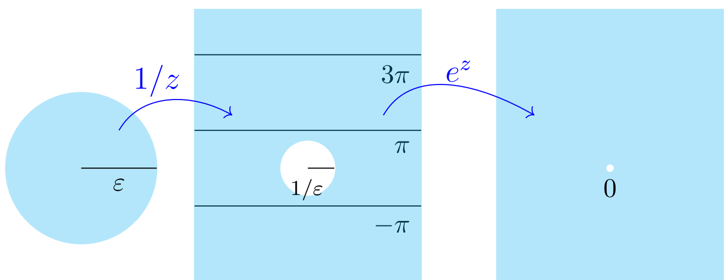
$$(z-\xi)^N f(z) = a_{-N} + a_{-N+1}(z-\xi) + \cdots + a_{-1}(z-\xi)^{N-1} + \varphi(z)(z-\xi)^N$$

with φ analytic at $z = \xi$. Hence, for $z \neq \xi$, we may write

$$f(z) = \frac{a_{-N}}{(z-\xi)^N} + \frac{a_{-N+1}}{(z-\xi)^{N-1}} + \cdots + \frac{a_{-1}}{z-\xi} + \varphi(z)$$

where $\displaystyle\sum_{k=1}^{N} \frac{a_{-k}}{(z-\xi)^k}$ is called the *singular part of f at ξ*.

Example $f(z) = e^{1/z}$ has an essential singularity at $\xi = 0$. Note that $f(D_\varepsilon(0) \setminus \{0\}) = \mathbb{C}^*$ for each $\varepsilon > 0$.



Note that the set $D_\varepsilon(0)\setminus\{0\}$ gets inverted outside the ball $D_{1/\varepsilon}(0)$ under the map $1/z$; and for each $n\in\mathbb{Z}$, the horizontal strip

$$H_n=\{z=x+iy \mid x\in\mathbb{R},\ \pi+2(n-1)\pi\le y<\pi+2n\pi\}$$

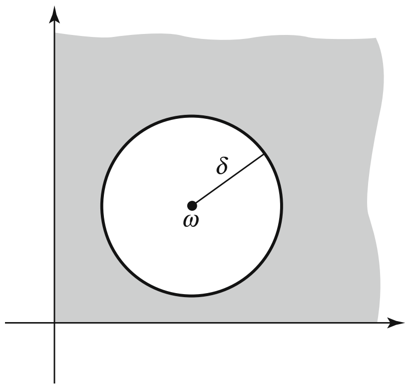
is mapped onto $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ by e^z. So, by choosing n, so that H_n lies outside $D_{1/\varepsilon}(0)$, we would have $f(D_\varepsilon(0)\setminus\{0\})=\mathbb{C}^*$. $\square$

The behavior of a function near an essential singularity is quite extreme, as illustrated by the following theorem.

Theorem 9. (Casorati-Weierstrass Theorem) If f has an essential singularity at ξ and if Δ' is a deleted open neighborhood of ξ, then $f(\Delta')=\{f(z)\mid z\in\Delta'\}$ is dense in $\mathbb{C}$ i.e. for any $\delta>0$ and for any $w\in\mathbb{C}$,

$$D_\delta(w)\cap f(\Delta')\neq\emptyset.$$

Remark Casorati-Weierstrass Theorem says that: an analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.



Proof Suppose not. Then $\exists\, D_\delta(w)$ with $D_\delta(w)\cap f(\Delta')=\emptyset$. This means that

$$|f(z)-w|>\delta \iff \frac{1}{|f(z)-w|}<\frac{1}{\delta}\quad \forall z\in\Delta'.$$

Thus $\dfrac{1}{f(z)-w}$ is bounded analytic on a deleted neighborhood Δ' of ξ which implies that ξ is a removable singularity of $\dfrac{1}{f(z)-w}$ and the function g defined by $g(z)=\dfrac{1}{f(z)-w}$ is analytic on the neighborhood $\Delta'\cup\{\xi\}$.

Since $f(z)=\dfrac{1+wg(z)}{g(z)}$ is a ratio of two analytic functions and by applying the Taylor Theorem on $g(z)$ at ξ, we conclude that

- ξ is a removable singularity of $f(z)$ if $g(\xi)\neq 0$,
- ξ is a pole of order N of $f(z)$ if $g^{(N)}(\xi)\neq 0$ and $g^{(k)}(\xi)=0$ for all $0\le k\le N-1$. $\square$

Suppose the statement is false, then $\exists\, z_0\in\mathbb{C}$ and $\delta>0$ such that

$$|f(z)-z_0|>\delta \quad \text{for all } z \text{ in a deleted neighborhood of } \xi \implies \lim_{z\to\xi}\frac{f(z)-z_0}{z-\xi}=\infty$$

and the function

$$g(z) := \frac{f(z) - z_0}{z - \xi} \quad \text{has a pole of order } N \text{ at } \; z = \xi,$$

so we may write $g(z) = (z-\xi)^{-N} g_N(z)$, where g_N is a nonzero analytic function in a neighborhood of ξ. Thus,

$$f(z) = (z-\xi)^{1-N} g_N(z) + z_0$$

which implies that

- if $N = 1$, then f has a removable singularity at $z = \xi$;
- if $N > 1$, then $f - z_0$, and f, have a pole of order $N - 1$ at $z = \xi$.

Hence, if f has an essential singularity at ξ, the statement of the theorem must be true. □

Definition We say f *meromorphic* in an open connected set Ω if f is analytic in Ω except at isolated poles.

3.3 The Local Mapping

Theorem 10. (Argument Principle I) Let f be analytic function in a disk $D_R(a)$ which does not vanish identically, and let $\{\zeta_j\}$ be the zeros of f, each zero being counted as many times as its order indicates. For every closed curve γ in $D_R(a)$ which does not pass through a zero, we have

$$\sum_j n(\gamma, \zeta_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz \tag{24}$$

Proof Case 1. f has a finite number of zeros $\{\zeta_j\}_{j=1}^n$ in $D_R(a)$.

For each $z \in D_R(a)$, by Taylor's formula, we may write

$$f(z) = (z-\zeta_1)\,(z-\zeta_2)\cdots(z-\zeta_n)\; g(z)$$

with g analytic and such that $g(z) \neq 0$, $\forall\, z \in D_R(a)$.

Hence, for any $z \in D_R(a)$ such that $z \neq \zeta_j$,

$$\frac{f'(z)}{f(z)} = \frac{1}{z-\zeta_1} + \frac{1}{z-\zeta_2} + \cdots + \frac{1}{z-\zeta_n} + \frac{g'(z)}{g(z)}$$

By Cauchy's theorem,

$$\int_\gamma \frac{g'(z)}{g(z)}\, dz = 0 \implies \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_j n(\gamma, \zeta_j) \tag{25}$$

Case 2. f has infinitely many zeros $\{\zeta_j\}_{j=1}^\infty$ in $D_R(a)$.

Since γ is inside $D_R(a)$, it is contained in a disk $D_{R'}(a)$ smaller than $D_R(a)$. Now, since f is not identically zero, it can only have finitely many zeros inside $D_{R'}(a)$, by the Bolzano-Weierstrass theorem and the identity theorem. Thus, the formula (25) holds inside $D_{R'}(a)$. It holds inside $D_R(a)$ as well since for the zeros of ζ_j of f outside of $D_{R'}(a)$, $n(\gamma, \zeta_j) = 0$. This concludes our proof. □

Remark

• Observe that the integral on the right of (24) can be represented as

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\,\gamma'(t)}{f(\gamma(t))}\,dt = \frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)'(t)}{(f\circ\gamma)(t)}\,dt = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dw}{w}$$

so, by letting $\Gamma = f\circ\gamma$, the equality (24) in the theorem can thus be interpreted as the equality

$$n(\Gamma,\,0) = \frac{1}{2\pi i}\int_{\Gamma}\frac{dw}{w} = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dw}{w} = \frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}\,dz = \sum_j n(\gamma,\,\zeta_j)$$

• The most useful application of the theorem is for the case when γ is a circle (or more generally a simple closed curve) so that

$$n(\gamma,\,\zeta_j) = \begin{cases} 0 & \text{if } \zeta_j \text{ is outside } \gamma \\ 1 & \text{if } \zeta_j \text{ is inside } \gamma \end{cases}$$

The formula in the theorem then gives a formula for the number of zeros enclosed by γ. This formula is at the heart of a number of numerical methods to locate the zeros of an analytic function.

• The name "argument principle" can be given the following intuitive-although not at all rigorous-interpretation:

$$\text{“}\ \frac{dw}{w} = d(\ln w) = d(\ln|w| + i\arg w)\ \text{”}$$

Note the quotes around these equalities, which should be seen as formal equalities and nothing else. For any curve that does not pass through 0, $\ln|w|$ is well defined, so by the fundamental theorem of calculus the contribution of the real part in the formal equalities above to the integral is zero when one integrates over a closed curve.

Open Mapping Theorem

Let $a\in\mathbb{C}$, and $\{\zeta_j(a)\}$ be the roots of the equation $f(z)=a$. Applying the argument principle theorem to the equation $f(z)=a$, we have

$$\sum_j n(\gamma,\,\zeta_j(a)) = \frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)-a}\,dz = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dw}{w-a} = n(\Gamma,\,a)$$

for any closed curve γ which does not pass through a point z such that $f(z)=a$.

Now, if two points a and b are both in the interior of $\Gamma = f\circ\gamma$, or both in its exterior,

$$n(\Gamma,\,a) = n(\Gamma,\,b) \iff \sum_j n(\gamma,\,\zeta_j(a)) = \sum_j n(\gamma,\,\zeta_j(b))$$

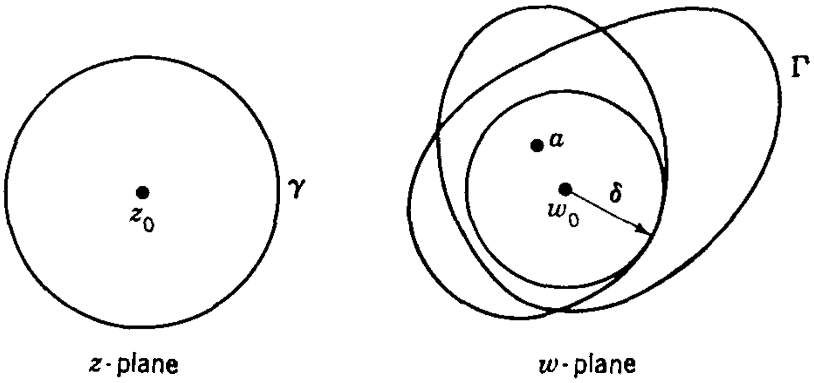
For the special case in which γ is a circle (or a simple closed curve), that means that f takes the values a and b an equal number of times inside γ. This leads to the following theorem.

Theorem 11. Suppose that f is analytic at z_0, and that $f(z)-w_0$ has a zero of order N at $z=z_0$. If $\varepsilon>0$ is sufficiently small, there exists $\delta>0$ such that $\forall\, a\in\mathbb{C}$ with $|a-w_0|<\delta$, the equation $f(z)=a$ has exactly N roots in the disk $|z-z_0|<\varepsilon$, i.e. $D_\delta(w_0)\subset f(D_\varepsilon(z_0))$ and each point in $D_\delta(w_0)$ is mapped exactly N times by points in $D_\varepsilon(z_0)$.

Proof Choose $\varepsilon>0$ such that

- f is analytic in $|z - z_0| \le \varepsilon$,
- z_0 is the only zero of $f(z) - w_0$ in the disk $|z - z_0| \le \varepsilon$,
- $f'(z) \ne 0$ for z such that $0 < |z - z_0| < \varepsilon$.

Let γ be the circle $|z - z_0| = \varepsilon$ and $\Gamma = f \circ \gamma$. Since Γ is closed and $w_0 = f(z_0) \notin \Gamma$, there exists a $\delta > 0$ such that $D_\delta(w_0) \cap \Gamma = \emptyset$. For each $a \in D_\delta(w_0)$, since



$$\sum_j n(\gamma, \zeta_j(a)) = n(\Gamma, a) = n(\Gamma, w_0) = \sum_j n(\gamma, \zeta_j(w_0)) = N,$$

the equation $f(z) = a$ has exactly N roots inside of γ, i.e. a is taken exactly N times in the disk $|z - z_0| < \varepsilon$. Furthermore, since ε is chosen small enough such that $f'(z) \ne 0$ for all $0 < |z - z_0| < \varepsilon$, the equation $f(z) = a$ has exactly N simple roots $\{\zeta_j(a)\}$ in the disk $|z - z_0| < \varepsilon$. □

Corollary 1. (Open Mapping Theorem) Let $\Omega \subseteq \mathbb{C}$ be an open connected set. If $f : \Omega \to \mathbb{C}$ is a nonconstant analytic function, then f maps open sets to open sets.

Proof For each $w_0 = f(z_0) \in f(\Omega)$ and any sufficiently small $\varepsilon > 0$, there exists a $\delta > 0$, as in the proof of Theorem 11, such that

$$D_\delta(w_0) \subseteq f(D_\varepsilon(z_0)) \subseteq f(\Omega)$$

which implies that w_0 is an interior point of $f(\Omega)$. Hence, f maps interior points are mapped to interior points, or equivalently, f maps open sets to open sets. □

Corollary 2. If f is analytic at z_0 and z_0 is a simple zero of $f(z) - w_0$, then there exists a neighborhood of z_0 and a corresponding neighborhood of w_0 on which f is one-to-one.

3.4 The Maximum Principle

Consider a function f which is analytic and nonconstant on an open connected set Ω. By the open mapping theorem, $\forall\, z_0 \in \Omega$, there exists an open disk $|w - f(z_0)| < \varepsilon$ contained in the image of Ω. In this open disk, there are points w such that $|w| > |f(z_0)|$. In other words, $|f(z_0)|$ is not the maximum value of $|f(z)|$. This proves the following :

Theorem 12. (Maximum Modulus Principle) If $f(z)$ is analytic and nonconstant in an open connected set Ω, then its modulus $|f(z)|$ has no maximum in Ω.

Remark The theorem is often reformulated in the following equivalent way:

Theorem 12′. If $f(z)$ is defined and continuous on a closed bounded set E, and analytic in the interior of E, then the maximum of $|f(z)|$ on E is assumed on the boundary of E.

Example Let $\Omega \subset \mathbb{C}$ be an open connected set such that $\Omega \cap \{0 + iy \mid y \in \mathbb{R}\} \neq \emptyset$. Show that there does not exist an analytic function $f(z) = f(x+iy)$ on Ω with modulus $|f(z)| = K/\cosh x$ for some constant $K \neq 0$.

Since $\cosh 0$ is a minimum of $\cosh x$, the modulus $|f(z)|$ achieves maximum at interior points $\{0 + iy \in \Omega\}$ of an open connected set Ω. Also since f cannot be constant by hypothesis, this would contradict the maximum modulus principle, so no such f exists. $\square$

Theorem 13. (The Lemma of Schwarz) If $f(z)$ is analytic in the disk $D_1(0) = \{z \mid |z| < 1\}$ and satisfies the conditions

$$f(0) = 0 \quad \text{and} \quad |f(z)| \leq 1 \quad \forall\, z \in D_1(0),$$

then $|f(z)| \leq |z|$ for all $z \in D_1(0)$ and $|f'(0)| \leq 1$. Furthermore, if $|f(z)| = |z|$ for some $z \in D_1(0) \setminus \{0\}$, or if $|f'(0)| = 1$, then $f(z) = Az$, with $A \in \mathbb{C}$ such that $|A| = 1$.

Proof For $z \in D_1(0)$, consider the function g defined by

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

From the hypotheses of the theorem, we know that g is analytic. For each $0 < R < 1$, since $|g(z)| \leq 1/R$ on the circle $C_R(0)$, we have

$$|g(z)| \leq \frac{1}{R} \quad \text{for all } z \in \overline{D}_R(0),\ 0 < R < 1$$

by the maximum modulus principle. Letting $R \to 1$, we conclude that $|g(z)| \leq 1$ for $|z| < 1$. This concludes the proof of the first part of the theorem.

Furthermore, if $|f(z)| = |z|$ holds for some $z \neq 0$, in $D_1(0)$, then g reaches its maximum modulus in the disk, so g is constant. The same reasoning holds in $|f'(0)| = 1$. $\square$

Remark The lemma of Schwarz is more powerful than one may at first think because its conditions can be generalized substantially:

- Consider an analytic function $f(z)$ on the unit disk, which maps $D_1(0)$ onto itself, and $z_0 \in D_1(0)$, with $f(z_0) = w_0$. The Möbius transformations

$$T(z) = \frac{z - z_0}{1 - \overline{z_0} z}$$

maps the unit disk onto itself, and is bijective. Likewise,

$$S(w) = \frac{w - w_0}{1 - \overline{w_0} w}$$

maps the unit disk onto itself. We conclude that the map $S \circ f \circ T^{-1}$ maps $D_1(0)$ onto itself, and $S\left(f\left(T^{-1}(0)\right)\right) = 0$. We can apply the lemma of Schwarz to $S \circ f \circ T^{-1}$, to obtain the inequality

$$\left| \frac{f\left(T^{-1}(w)\right) - f(z_0)}{1 - \overline{f(z_0)} f\left(T^{-1}(w)\right)} \right| = \left| \frac{f\left(T^{-1}(w)\right) - w_0}{1 - \overline{w_0} f\left(T^{-1}(w)\right)} \right| = \left| S(f\left(T^{-1}(w)\right)) \right| \leq |w|$$

With $z = T^{-1}(w) \iff w = T(z)$, this can be rewritten as

$$\left| \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)} \right| \leq \left| \frac{z - z_0}{1 - \overline{z_0} z} \right| \quad , \quad \forall\, z,\, z_0 \in D_1(0)$$

• The lemma of Schwarz can be generalized further to functions with upper bound $M \in \mathbb{R}_+$ instead of 1 : we then apply the lemma to $\dfrac{f(z)}{M}$, and M will appear on the right hand side of the inequality.

• Likewise, if f satisfies the conditions on a disk of radius R instead of the unit disk, we apply the theorem to $f(Rz)$, and $\dfrac{1}{R}$ will appear on the right-hand side of the inequality.

Example Let $a, b, c, d \in \mathbb{C}$ such that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$, and $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be a linear fractional transformation (or bilinear transformation, Möbius transformation) defined as follows.

$$\text{If } c \neq 0,\ T(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } z \in \mathbb{C} \setminus \{-d/c\} \\ \dfrac{a}{c} & \text{if } z = \infty \\ \infty & \text{if } z = -d/c \end{cases} \quad ; \quad \text{if } c = 0,\ T(z) = \begin{cases} \dfrac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases}$$

Note that

$$\text{if } c \neq 0,\ z \in \mathbb{C} \setminus \{-d/c\},\ T(z) = \frac{az+b}{cz+d} = \frac{1}{c}\left[a - \left(\frac{ad-bc}{cz+d}\right)\right] = T_3 \circ T_2 \circ T_1(z)$$

$$\text{if } c = 0,\ z \in \mathbb{C},\ T(z) = \frac{1}{d}(az+b)$$

are composition of maps T_i, $i = 1, 2, 3$, where

$$\begin{aligned} T_1(z) &= cz + d \quad \text{a dilation and a translation} \\ T_2(z) &= \frac{1}{z} \quad \text{an inversion} \\ T_3(z) &= \frac{a}{c} - \left(\frac{ad-bc}{c}\right) z \quad \text{a dilation and a translation} \end{aligned}$$

and each T_i maps {circles and lines} to {circles and lines}, so

- T : {circles and lines} $\to$ {circles and lines}
- T is bijective and invertible such that

$$T(z) = \frac{az+b}{cz+d} \quad \Longleftrightarrow \quad T^{-1}(z) = \frac{dz-b}{-cz+a}. \quad \square$$

Example For each $a \in \mathbb{C}$ such that $|a| < 1$, let $B_a(z) : \bar{D}_1(0) \to \mathbb{C}$ be the bilinear transformation defined by

$$B_a(z) = \frac{z-a}{1-\bar{a}z} \quad \text{for } |z| \leq 1.$$

This implies that $B_0(z) = z$, and for $0 < |a| < 1$, we have

- $B_a(z)$ is analytic throughout $|z| \leq 1$ since $\dfrac{1}{|\bar{a}|} = \dfrac{1}{|a|} > 1$,

- $B_a(z)$ maps the unit circle $|z|=1$ onto $|z|=1$ since

$$|B_a(z)|^2 = \frac{z-a}{1-\bar{a}z}\,\frac{\bar{z}-\bar{a}}{1-a\bar{z}} = \frac{|z|^2 - a\bar{z} - \bar{a}z + |a|^2}{1 - a\bar{z} - \bar{a}z + |a|^2|z|^2} = 1 \quad \text{for } |z|=1,$$

  this implies that $B_a(z) : \bar{D}_1(0) \to \bar{D}_1(0)$, i.e. $|B_a(z)| \le 1$ for all $z \in \bar{D}_1(0)$.

- $B_a(a) = 0,\ B_a'(z) = \dfrac{1-|a|^2}{(1-\bar{a}z)^2} \implies B_a'(a) = \dfrac{1}{1-|a|^2}$ and $B_a^{-1}(z) = \dfrac{z+a}{1+\bar{a}z} = B_{-a}(z)$. $\square$

Corollary (Generalizied Schwarz Lemma) If $f : D_1(0) \to D_1(0)$ is analytic (extending continuously to the boundary) and $f(a)=0$ for some $|a|<1$, then

(i) $|f(z)| \le |B_a(z)|$ for all $z \in D_1(0)$,

(ii) $|f'(a)| \le |B_a'(a)| = \dfrac{1}{1-|a|^2}$.

and equality holds in either (i) or (ii) if and only if $f(z) = e^{i\theta}B_a(z)$ for some $\theta \in \mathbb{R}$.

Proof Let $g : D_1(0) \to \mathbb{C}$ be defined by

$$g(z) = \begin{cases} \dfrac{f(z)}{B_a(z)} = \dfrac{f(z)-f(a)}{B_a(z)-B_a(a)} & \text{if } z \ne a, \\[2ex] \dfrac{f'(a)}{B_a'(a)} = f'(a)(1-|a|^2) & \text{if } z = a. \end{cases}$$

For each $0<r<1$, since g is analytic, $|f(z)| \le 1$ for all $z \in D_1(0)$ and since $|B_a(z)| = 1$ for all $|z|=1$, we have, by the Maximum Modulus Theorem,

$$\max_{|z|\le r} |g(z)| = \max_{|z|=r} |g(z)| = \max_{|z|=r} \frac{|f(z)|}{|B_a(z)|} \le \max_{|z|=r} \frac{1}{|B_a(z)|}.$$

By letting $r \to 1$, we obtain that

$$|g(z)| \le \lim_{r\to 1} \max_{|z|=r} \frac{1}{|B_a(z)|} = 1 \quad \forall\, |z| \le 1$$

which implies that

$$\begin{cases} |f(z)| & \le |B_a(z)| \quad \forall\, z \in D_1(0) \\ |f'(a)| & \le |B_a'(a)| = \dfrac{1}{1-|a|^2} \end{cases}$$

If the equality occurs in either case, i.e.

- either $|g(z_0)| = 1$ for some $z_0 \in D_1(0),\ z_0 \ne a$,
- or $|g(a)| = |f'(a)| = \dfrac{1}{1-|a|^2}$,

then g is a constant function on $D_1(0)$ with modulus equals to 1, and there exists a $\theta \in \mathbb{R}$ such that $g(z) = e^{i\theta}\ \forall z \in D_1(0)$ which implies that

$$f(z) = e^{i\theta}B_a(z)\ \forall z \in D_1(0) \setminus \{a\} \implies f(z) = e^{i\theta}B_a(z)\ \forall z \in D_1(0) \text{ since } f(a)=0. \quad \square$$

Example Let $H = \{f \mid f : D_1(0) \to D_1(0) \text{ is anaytic}\}$. Find $\max\limits_{f\in H} |f'(\tfrac{1}{3})|$.

Case 1 Suppose $f(\frac{1}{3}) = 0$, we have

$$|f'(\tfrac{1}{3})| \le |B'_{1/3}(\tfrac{1}{3})| \quad \text{by the Generalized Schwarz Lemma.}$$

Case 2 Suppose $f(\frac{1}{3}) \neq 0$, we consider the map g defined by

$$g(z) = \frac{f(z) - f(\frac{1}{3})}{1 - \overline{f(\frac{1}{3})} f(z)} = B_{f(1/3)}(f(z)) \quad \text{for } |z| \le 1.$$

Since $|f(z)| < 1$ is analytic for all $|z| < 1$, and $B_{f(1/3)}(w)$ is analytic with $|B_{f(1/3)}(w)| < 1$ for all $|w| < 1$,

$$g(z) = B_{f(1/3)}(f(z)) : D_1(0) \to D_1(0) \quad \text{is analytic on } D_1(0) \text{ with } g(1/3) = 0.$$

Thus, by **Case** 1, we have

$$|g'(\tfrac{1}{3})| \le |B'_{1/3}(\tfrac{1}{3})|.$$

Since

$$g'(1/3) = \frac{f'(\frac{1}{3})}{1 - \left|f(\frac{1}{3})\right|^2} \quad \text{and} \quad 0 < 1 - \left|f(\tfrac{1}{3})\right|^2 < 1$$

this implies that

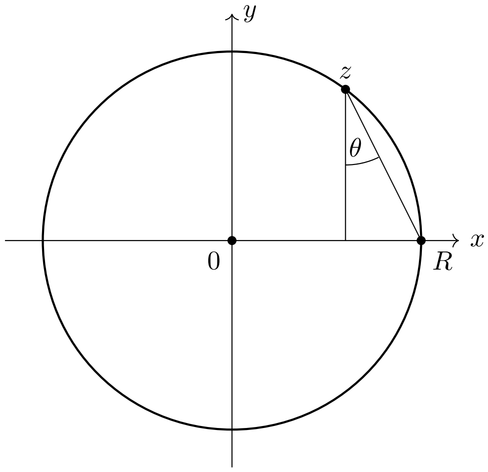
$$\left|f'(\tfrac{1}{3})\right| < |g'(\tfrac{1}{3})| \le |B'_{1/3}(\tfrac{1}{3})| \implies \max_{f \in H} |f'(\tfrac{1}{3})| = |B'_{1/3}(\tfrac{1}{3})| = \frac{1}{1 - (1/3)^2} = \frac{9}{8}. \quad \square$$

Example If f is entire satisfying

$$|f(z)| \le \frac{1}{|\operatorname{Im} z|} \quad \forall z \in \mathbb{C} \setminus \mathbb{R},$$

then $f \equiv 0$.

Proof



For any $R > 0$ and for each $z \in \mathbb{C} \setminus \mathbb{R}$ satisfying that $|z| = R$, note that if

$$\operatorname{Re} z \ge 0 \implies |(z - R) f(z)| \le \frac{|z - R|}{|\operatorname{Im} z|} = \sec\theta \le \sqrt{2} \quad \text{for some } \theta \in [0, \frac{\pi}{4}],$$

$$\operatorname{Re} z \le 0 \implies |(z + R) f(z)| \le \frac{|z + R|}{|\operatorname{Im} z|} = \sec\theta \le \sqrt{2} \quad \text{for some } \theta \in [0, \frac{\pi}{4}].$$

Thus the entire function g defined by $g(z) = (z^2 - R^2)f(z)$ satisfies that

$$|g(z)| = |z+R|\,|z-R|\,|f(z)| \le 3R \quad \forall\, z \in \mathbb{C} \text{ with } |z| = R.$$

By the Maximum-Modulus Theorem,

$$|g(z)| = |z^2 - R^2|\,|f(z)| \le 3R \quad \forall\, |z| \le R \implies |f(z)| \le \frac{3R}{|z^2 - R^2|} \quad \forall\, |z| \le R,\ \forall R > 0.$$

By letting $R \to \infty$, we obtain that $f(z) = 0$ for each (fixed) $z \in \mathbb{C}$. $\square$

§4 The General form of Cauchy's Theorem

4.1 Chains and Cycles

Definition Let Ω be an open set in $\mathbb{C}$. A *chain in Ω* is a finite collection $\gamma_j : [a_j, b_j] \to \Omega$, $j = 1, \dots, N$ of piecewise continuously differentiable curves in Ω.

Writing $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_N$ for a given chain, we can integrate a continuous function f in Ω along Γ as follows:

$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{N} \int_{\gamma_j} f(z)\,dz$$

A *cycle in Ω* is a chain $\Gamma = \sum_{j=1}^{N} \gamma_j$ where each point $z \in \mathbb{C}$ is an initial point of just as many of the γ_j as it is a terminal point. In other words, a cycle is a finite sum of closed curves. As an illustration, the *index* of a point z with respect to the cycle Γ is

$$n(\Gamma, z) = \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\gamma_j} \frac{d\zeta}{\zeta - z}$$

Observe that the integrals in the sum on the right-hand side of the last equality above may not be over closed curves.

4.2 Simple Connectivity

Below, we start this section with an unusual definition for simple connectedness. Its weakness is that it is not general, in the sense that it cannot be used in $\mathbb{R}^n$ with $n \ge 3$. However, we will show later in this course that for $\mathbb{C}$, it is equivalent to the more common definition, which says that any simple closed curve can be shrunk to a point continuously in the set. And the advantage of our unusual definition is that it is more convenient for the proof of the general form of Cauchy's theorem.

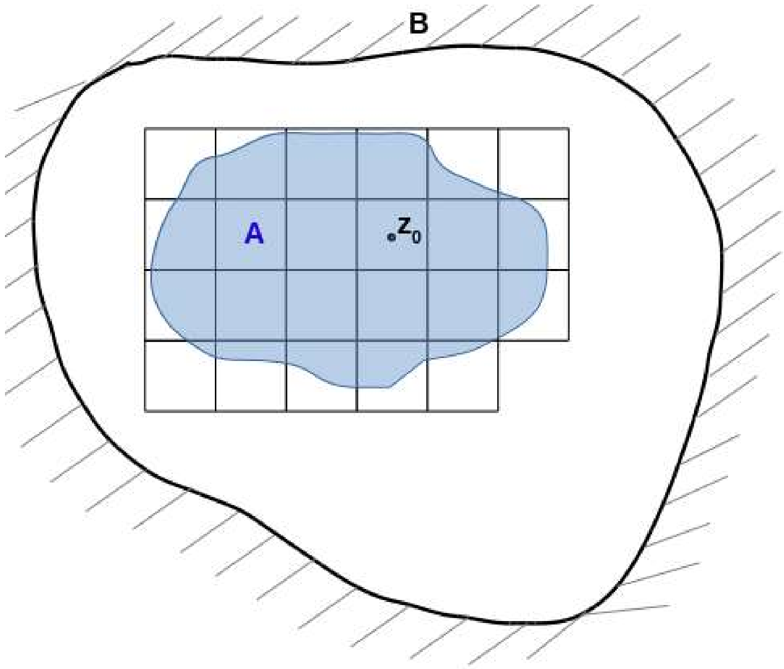
Definition An open connected set $\Omega \subset \mathbb{C}$ is said to be *simply connected* if its complement with respect to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is connected, i.e. Ω is simply connected if $\widehat{\mathbb{C}} \setminus \Omega$ is connected.

Theorem 14. An open connected set $\Omega \subset \mathbb{C}$ is simply connected if and only if $n(\gamma, z) = 0$ for all cycles γ in Ω and all points $z \notin \Omega$.

Proof

($\Longrightarrow$) For any cycle $\gamma \in \Omega$, since $\Omega \subset \mathbb{C}$ is simply connected, $\widehat{\mathbb{C}} \setminus \Omega$ is connected and must be in one of the (interior or exterior) regions determined by γ. Now $\infty \in \widehat{\mathbb{C}} \setminus \Omega$, the set $\widehat{\mathbb{C}} \setminus \Omega$ must be the unbounded region defined by γ. Since $\lim_{|z|\to\infty} n(\gamma, z) = 0$ and $n(\gamma, z) : \mathbb{C}\setminus\{\gamma\} \to \mathbb{Z}$ is continuous, we have $n(\gamma, z) = 0\ \forall\, z \notin \Omega$.

($\Longleftarrow$) Suppose that Ω is not simply connected, and the complement set $\widehat{\mathbb{C}}\setminus\Omega = A\cup B$ with A and B disjoint closed sets, with a shortest distance $\delta > 0$ between the two sets. Let us say that B is the unbounded set, so A is bounded (and compact). We cover A with a net of squares S whose sides have length $\ell < \delta/\sqrt{2}$, constructed in such a way that $z_0 \in A$ (so $z_0 \notin \Omega$) lies at the center of a square (as shown in the figure).



Since A is compact, there exists a finite collection of squares $\{S_j\}_{j=1}^n$ such that $A \subset \bigcup_{j=1}^{n} \operatorname{Int} S_j$. Let $\gamma = \sum_{j=1}^{n} \partial S_j$, where ∂S_j is the boundary curve of square S_j, and observe first that $n(\gamma, z_0) = 1$ since z_0 belongs to only one of the squares $\{S_j\}_{j=1}^n$.

Furthermore, it is clear that $\gamma \cap B = \emptyset$. Now, since $A \subset \bigcup_{j=1}^{n} \operatorname{Int} S_j$, the curve $\widetilde{\gamma} = \partial\left(\bigcup_{j=1}^{n} S_j\right)$ is a cycle such that $\widetilde{\gamma} \subset \gamma \implies \widetilde{\gamma}\cap B = \emptyset$, $\widetilde{\gamma}\cap A = \emptyset$, $\widetilde{\gamma}$ is a cycle in Ω and since the integral corresponding to $n(\gamma, z_0)$, all the sides of the squares contained in A are traversed exactly twice, in opposite directions, and therefore cancel, so we have

$$n(\widetilde{\gamma}, z_0) = n(\gamma, z_0) = 1 \quad \text{for some } \widetilde{\gamma} \subset \Omega = \widehat{\mathbb{C}}\setminus(A\cup B), \text{ and } z_0 \in A \text{ (so } z_0 \notin \Omega) \quad \square$$

4.3 Homology

Definition A cycle γ in an open set Ω is said to be *homologous to zero* with respect to Ω if $n(\gamma, z) = 0$ for all z in the complement of Ω in $\widehat{\mathbb{C}}$.

Remark We write $\gamma \sim 0 \pmod{\Omega}$, or simply $\gamma \sim 0$, if the cycle γ is homologous to zero with respect to Ω, and write $\gamma_1 \sim \gamma_2$ when $\gamma_1 - \gamma_0 \sim 0$. Note that with this notation, the Theorem 14 can be written as follows.

Theorem An open connected set $\Omega \subset \mathbb{C}$ is simply connected if and only if $\gamma \sim 0$ for all γ in Ω.

4.4 The General Statement of Cauchy's Theorem

We now have all the tools required to give Cauchy's theorem in its most general form.

Theorem 15. (Cauchy's Theorem) If f is analytic in the open set Ω, then $\int_\gamma f(z)\,dz = 0$ for every cycle γ which is homologous to zero in Ω.

Proof Consider γ such that $\gamma \sim 0 \pmod{\Omega}$, and the set

$$E = \{z \in \mathbb{C} \setminus \{\gamma\} \mid n(\gamma, z) = 0\}$$

which is open since $n(\gamma, z) : \mathbb{C} \setminus \{\gamma\} \to \mathbb{Z}$ is continuous and $\{0\}$ is open in $\mathbb{Z}$.

Let $g : \Omega \times \Omega \to \mathbb{C}$ be defined by

$$g(z, \zeta) := \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \text{if } z \neq \zeta \\ f'(z) & \text{if } z = \zeta \end{cases}$$

Then g is continuous in both its variables. Furthermore, for each $\zeta_0 \in \Omega$, the function $\widetilde{g} : \Omega \to \mathbb{C}$ defined by $\widetilde{g}(z) = g(z, \zeta_0)$ is analytic in Ω since $\lim\limits_{z \to \zeta_0} (z - \zeta_0)\,\widetilde{g}(z) = \lim\limits_{z \to \zeta_0} (z - \zeta_0)\,g(z, \zeta_0) = 0$, so ζ_0 is a removable singularity of $\widetilde{g}$.

Consider the function h on $\mathbb{C}$ defined by

$$h(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_\gamma g(z, \zeta)\,d\zeta & \text{if } z \in \Omega \\ \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta & \text{if } z \in E \end{cases}$$

Since $\mathbb{C} \setminus \Omega \subset E$, and

$$\int_\gamma g(z, \zeta)\,d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - f(z) n(\gamma, z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad \text{for all } z \in \Omega \cap E,$$

we have $\Omega \cup E = \mathbb{C}$ and the two definitions of h agree on $\Omega \cap E$, so h is defined on all of $\mathbb{C}$.

Also since h is analytic in E by Lemma 3 of 4.2.3, the two definitions of h agree on $\Omega \cap E$ and $\Omega \cup E = \mathbb{C}$, h is entire if we can prove that h is analytic on Ω.

Lemma Let $[a, b] \subset \mathbb{R}$, Ω be an open connected set in $\mathbb{C}$. Suppose that $\varphi : \Omega \times [a, b] \to \mathbb{C}$ is continuous, and the function $z \mapsto \varphi(z, t)$ is analytic on Ω for each $t \in [a, b]$. Then the function $F : \Omega \to \mathbb{C}$ defined by

$$F(z) = \int_a^b \varphi(z, t)\,dt$$

is analytic on Ω.

Proof Let $z_0 \in \Omega$, and let $R > 0$ be such that $D_R(z_0) \subset \Omega$.

For each $z \in D_R(z_0)$, since $\varphi(z, t)$ is analytic on Ω for each $t \in [a, b]$ and by the Cauchy integral formula, we have

$$\begin{aligned} F(z) = \int_a^b \varphi(z, t)\,dt &= \frac{1}{2\pi i} \int_a^b \left(\int_{|\zeta - z_0| = R} \frac{\varphi(\zeta, t)}{\zeta - z}\,d\zeta \right) dt \\ &= \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \left(\int_a^b \varphi(\zeta, t)\,dt \right) \frac{d\zeta}{\zeta - z} \end{aligned}$$

where the last equality holds by the Fubini's Theorem. Now, $\int_a^b \varphi(\zeta, t)\, dt$ is a continuous function of ζ, F is analytic on $D_R(z_0)$ by Lemma 3 of 4.2.3. $\square$

Proof of Theorem 15 **(cont'd)** We conclude that h is entire. Now, for $|z|$ sufficiently large, since $n(\gamma, z) = 0$, we have $z \in E$ and thus

$$h(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Also, since f is bounded on γ, $\lim_{|z|\to\infty} h(z) = 0$, we conclude that h is bounded, and $h \equiv 0$ by Liouville's theorem.

Hence, for each $z \in \Omega \setminus \gamma$, we have

$$\frac{1}{2\pi i} \int_\gamma g(z, \zeta)\, d\zeta = 0 \iff n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

This is the generalized version of Cauchy's integral formula, which we can now use to prove Theorem 15 (Cauchy's Theorem).

Let $z_0 \in \Omega \setminus \gamma$, and let $F(z) = (z - z_0) f(z)$ for each $z \in \Omega$. Then

$$\int_\gamma f(z)\, dz = \int_\gamma \frac{F(z)}{z - z_0}\, dz = 2\pi i\, n(\gamma, z_0) F(z_0) = 0$$

This completes this very elegant proof, first proposed by John Dixon in the *Proceedings of the American Mathematical Society*, Volume 29, Number 3, August 1971. $\square$

Corollary 1. If f is analytic in a simply connected open set Ω, then $\int_\gamma f(z)\, dz = 0$ for all cycles in Ω.

Proof This follows directly from Cauchy's Theorem (Theorem 15). $\square$

Corollary 2. If f is analytic and nonzero in a simply connected open region Ω, then it is possible to define single-valued analytic branches of $\ln\,[f(z)]$ and $\sqrt[n]{f(z)}$ in Ω.

Proof Since f is analytic and nonzero, $f'(z)/f(z)$ is analytic and

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0 \quad \text{for all cycles in } \Omega \text{ by Cauchy's theorem.}$$

By the Fundamental Theorem of Calculus, there exists an analytic function F (called a primitive of f) such that

$$\begin{aligned} F'(z) = \frac{f'(z)}{f(z)}\ \forall\, z \in \Omega &\iff \frac{d}{dz}\left[f(z) e^{-F(z)}\right] = 0\ \forall\, z \in \Omega \\ &\iff f(z) = A e^{F(z)}\ \forall\, z \in \Omega,\ \text{for some } A \in \mathbb{C} \setminus \{0\} \end{aligned}$$

Choose $z_0 \in \Omega$ and one of the infinitely many values of $\ln\,[f(z_0)]$. Since

$$\exp\,[F(z) - F(z_0) + \ln\,[f(z_0)]] = \frac{f(z)}{A}\, e^{-F(z_0)} f(z_0) = f(z)$$

we can therefore define a single-valued, analytic branch of the logarithm of f as

$$\ln f(z) = F(z) - F(z_0) + \ln f(z_0)$$

The definition of $\sqrt[n]{f}$ follows from this result, as $\forall\, z \in \Omega$ we write $\sqrt[n]{f} = \exp\left[\frac{1}{n} \ln\,(f(z))\right]$. $\square$

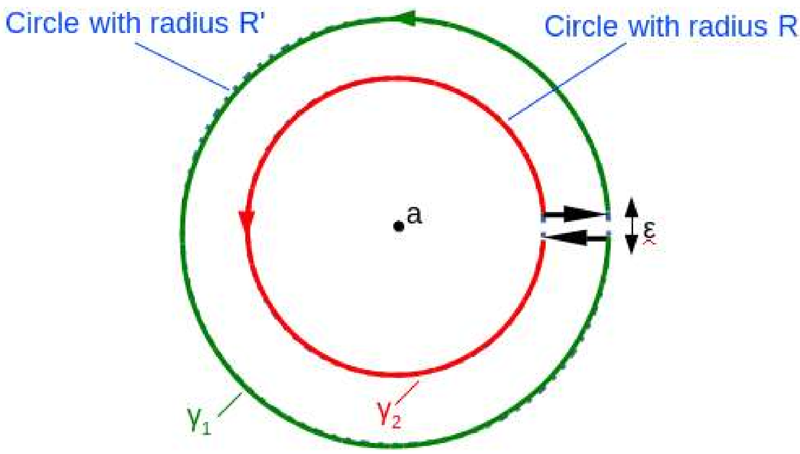
§ 5 The Calculus of Residues

5.1 The Residue Theorem

Definition Let Ω be an open connected set in $\mathbb{C}$, a be a point in Ω, f be an analytic function in $\Omega \setminus \{a\}$, and $C_R(a)$ be a circle in Ω with center a. *The residue of f at a*, denoted $\mathrm{Res}_{z=a} f(z)$, is defined by

$$\mathrm{Res}_{z=a} f(z) = \frac{1}{2\pi i}\int_{C_R(a)} f(z)\,dz$$

Remark Note that this definition is independent of the choice of the radius $R > 0$ of the circle C. For example, if $C_{R'}(a)$ is a circle centered in a and contained in Ω, and is γ the cycle made of the piecewise differentiable green, red and black arcs shown in the figure,



then, by the general form of Cauchy's Theorem (Theorem 15), we have

$$\int_\gamma f(z)\,dz = 0 \iff \int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz + I_\varepsilon$$

where I_ε is the contribution from the two black horizontal segments separated by a distance ε. Since f is continuous in $\Omega \setminus \{a\}$, and $\lim\limits_{\varepsilon\to 0} I_\varepsilon = 0$, we have

$$\int_{C_{R'}(a)} f(z)\,dz = \lim_{\varepsilon\to 0}\int_{\gamma_1} f(z)\,dz = \lim_{\varepsilon\to 0}\left(\int_{\gamma_2} f(z)\,dz + I_\varepsilon\right) = \int_{C_R(a)} f(z)\,dz$$

In general, if f is analytic in the open connected set Ω except for finitely many singularities a_j, and if γ is a cycle in $\Omega' = \Omega \setminus \{a_j\}_{j=1,\ldots,N}$ which is homologous to zero with respect to Ω, then $\gamma \sim \sum\limits_{j=1}^{N} n(\gamma, a_j)C_j \pmod{\Omega'}$, where C_j is any circle in Ω' with center a_j, and, by the general formulation of Cauchy's theorem, we have

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{N} n(\gamma, a_j)\int_{C_j} f(z)\,dz \iff \frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{j=1}^{N} n(\gamma, a_j)\,\mathrm{Res}_{z=a_j} f(z) \quad (26)$$

The result can naturally be extended to the case in which f has infinitely many singularities, as we have done in 4.3.3. The sum in (26) is always finite, and known as the following theorem.

Theorem 17. **(Residue Theorem)** Let f be analytic except for isolated singularities a_j in an open connected set Ω. Then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_j n(\gamma,\, a_j)\,\mathrm{Res}_{z=a_j} f(z) \tag{27}$$

for any cycle γ which is homologous to zero in Ω and does not pass through any of the points a_j, and the sum (27) is finite.

Remark As one may expect, the residue theorem is particularly convenient to use when γ is such that $\forall\, a_j,\ n(\gamma,\, a_j) = 0$ or 1.

More importantly, it is only useful as a tool for integration if there is a simple method to compute residues. When f has essential singularities, such a method is not available, and residue calculus is not particularly useful.

However, *if f has a pole of order N at a,* then $g(z) = (z-a)^N f(z)$ is analytic in a neighborhood of a. Integrating along a circle C centered in a and in that neighborhood, we may write

$$g^{(N-1)}(a) = \frac{(N-1)!}{2\pi i}\int_C \frac{g(z)}{(z-a)^N}\,dz = (N-1)!\,\mathrm{Res}_{z=a} f(z)$$

Hence,

$$\mathrm{Res}_{z=a} f(z) = \frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}\left[(z-a)^N f(z)\right]\Big|_{z=a} \tag{28}$$

In particular, if $f(z) = \dfrac{g(z)}{h(z)}$, h has a simple zero at a and $g(a) \neq 0$, then f has a simple pole at a with

$$\mathrm{Res}_{z=a} f(z) = \lim_{z\to a}\frac{(z-a)g(z)}{h(z)} = \frac{g(a)}{h'(a)}$$

Example Use the residue theorem to compute

$$\oint_{|z|=1} \frac{e^{iz}}{z^2}\,dz$$

where the circle is traversed in the counterclockwise direction.

5.2 The Argument Principle

Theorem 18. **(Argument Principle II)** If f is meromorphic in an open connected set Ω, with zeros a_j and poles b_k, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_j n(\gamma,\, a_j) - \sum_k n(\gamma,\, b_k) \tag{29}$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros and poles. The sums in (29) are finite, and multiple zeros and poles have to be repeated as many times as their order indicates, i.e. if a_j is a zero of order $N_j > 0$, then the term $n(\gamma,\, a_j)$ is repeated N_j times and if b_k is a zero of order $N_k < 0$, then the term $n(\gamma,\, b_k)$ is repeated $-N_k$ times in the sum.

Proof Let us first assume that the function has a finite number of zeros and poles, and call K that number. Consider the orders N_j of the zeros and poles z_j of f in Ω. $N_j > 0$ if z_j is a zero of f, $N_j < 0$ if z_j is a pole of f. Let

$$g(z) := f(z) \prod_{j=1}^{K} (z - z_j)^{-N_j}$$

Note that g only has removable singularities in Ω, so we can view it as analytic in Ω. Furthermore, g does not have zeros inside Ω. Writing $f(z) = g(z) \prod_{j=1}^{K} (z - z_j)^{N_j}$ and taking the logarithmic derivative of that equality for $z \neq z_j$, we find

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{K} \frac{N_j}{z - z_j} + \frac{g'(z)}{g(z)}$$

Integrating this equality along any cycle γ which is homologous to zero in Ω and does not pass through any z_j, we get $\int_\gamma \frac{g'(z)}{g(z)}\, dz = 0$ by Cauchy's Theorem, and

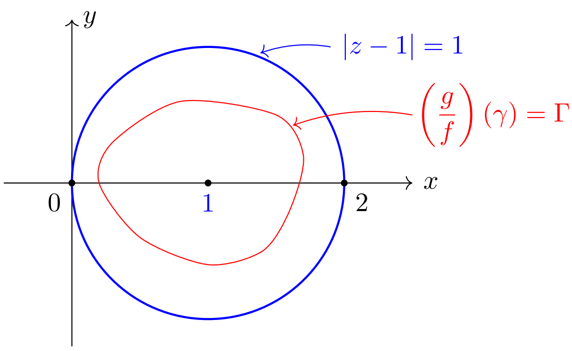
$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \sum_{j=1}^{K} \frac{N_j}{z - z_j}\, dz = \sum_{j=1}^{K} n(\gamma, z_j) N_j,$$

by the definition of the index of z_j with respect to γ (or by the residue theorem). $\square$

Remark The proof can be extended to the situation in which the function f may have an infinite number of zeros and/or poles, using the same method as we did in 4.3.3 to show that the formula in (29) remains true, with the sums still finite.

Corollary (Rouché's Theorem) Let γ be a cycle which is homologous to zero in the open connected set Ω and such that $n(\gamma, z)$ is either 0 or 1 for all $z \in \Omega$ such that $z \notin \gamma$. Suppose that f and g are analytic in Ω, and that $|f(z) - g(z)| < |f(z)|$ for all $z \in \gamma$. Then f and g have the same number of zeros enclosed by γ.

Proof Since $|f(z) - g(z)| < |f(z)|$ for all $z \in \gamma$, $f(z) \neq 0$ and $g(z) \neq 0$ for all $z \in \gamma$, and $\psi(z) := \dfrac{g(z)}{f(z)}$ satisfies that $|\psi(z) - 1| < 1$ for all $z \in \gamma$.



Hence,

$$\int_\gamma \frac{\psi'(z)}{\psi(z)}\, dz = \int_\Gamma \frac{d\zeta}{\zeta} = 2\pi i\, n(\Gamma, 0) = 0 \quad \text{where } \zeta = \psi(z) \text{ , and } \Gamma = \psi(\gamma)$$

Now, let N_g be the number of zeros of g inside γ, and N_f the number of zeros of f inside γ. By the argument principle,

$$0 = \int_\gamma \frac{\psi'(z)}{\psi(z)}\, dz = N_g - N_f \iff N_f = N_g \quad \square$$

Example Consider the polynomial $z^4 - 6z + 3$. How many zeros does it have in the annulus between $|z| = 1$ and $|z| = 2$?

Start with $\gamma_1 : |z| = 2$, and take $f_1(z) = z^4$, $g_1(z) = z^4 - 6z + 3$.

$$\forall\, z \in \gamma_1 \ , \ |f_1(z) - g_1(z)| = |6z - 3| \leq 15 < 16 = |f_1(z)|$$

Hence both $f_1(z)$ and $g_1(z) = z^4 - 6z + 3$ have 4 zeros inside $|z| = 2$.

Now consider $\gamma_2 : |z| = 1$, and define $f_2(z) = -6z$, $g_2(z) = z^4 - 6z + 3$.

$$\forall\, z \in \gamma_2 \ , \ |f_2(z) - g_2(z)| = |z^4 + 3| \leq 4 < 6 = |f_2(z)|$$

So both $f_2(z)$ and $g_2(z) = z^4 - 6z + 3$ have 1 zero inside $|z| = 1$.

We conclude that $z^4 - 6z + 3 = 0$ has 3 roots in the annulus. $\square$

§ 6 Harmonic Functions

6.1 Definition and Basic Properties

On several occasions in this course we pointed out close links between results obtained for analytic functions and results concerning harmonic functions we may already know from courses on Partial Differential Equations. The purpose of this lecture is to give these links a rigorous background.

Definition A function $u : (x, y) \in \Omega \to \mathbb{R}$ is *harmonic* in Ω if $u \in C^2(\Omega)$ and satisfies Laplace's equation

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad \forall\, (x, y) \in \Omega$$

Remark Note that

- if $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic in Ω, then u and v satisfy the Cauchy-Riemann equations, and are therefore harmonic in Ω.
- if u is harmonic in Ω, then $g(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ is analytic in Ω, since the real and imaginary parts of g satisfy the Cauchy-Riemann equations.
- $u(x, y) = \ln\left(\sqrt{x^2 + y^2}\right)$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$ without a single-valued conjugate function.

Question Under which conditions does a harmonic function u on Ω have a *harmonic conjugate* $v : \Omega \to \mathbb{R}$ such that $f = u + iv$ is analytic on Ω ?

Definition Let $u : (x, y) \in \Omega \to \mathbb{R}$ be a smooth function. The *conjugate differential of u*, denoted $\star du$, is defined by

$$\star du := -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy$$

where the differential $du = \dfrac{\partial u}{\partial x}\, dx + \dfrac{\partial u}{\partial y}\, dy$, and $\star$ is called the *Hodge $\star$-operator*.

Remark Observe that if u is harmonic function in Ω, then $f(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ is analytic in Ω, and we may write the differential

$$f\,dz = \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right) + i\left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\right) = du + i \star du$$

Lemma Let Ω be an open set in $\mathbb{C}$. If u is harmonic function in Ω, then $\displaystyle\int_\gamma \star du = 0$ for every cycle γ homologous to zero in Ω.

Proof If u is harmonic in Ω, since $f(z) = u_x - iu_y$ is analytic in Ω, $f\,dz = du + i \star du$,

- $\displaystyle\int_\gamma f(z)\,dz = 0$ for every cycle γ homologous to zero in Ω by Cauchy's theorem,
- $\displaystyle\int_\gamma du = 0$ for every cycle γ in Ω by the Fundamental Theorem of Calculus,

we have

$$0 = \int_\gamma f(z)\,dz = \int_\gamma du + i\int_\gamma \star du = i\int_\gamma \star du \implies \int_\gamma \star du = 0 \quad \forall\,\gamma \sim 0 \pmod{\Omega}. \quad \square$$

Theorem In a simply connected open set Ω, any harmonic function u has a single-valued conjugate function v which is uniquely determined up to an additive constant.

Proof Existence Since Ω is simply connected, and $\displaystyle\int_\gamma \star du = 0$ for all cycles γ in Ω, so, by the Independence of Path Theorem (4.1.3 Theorem 1), $\star du = -u_y dx + u_x dy$ is an exact differential on Ω, i.e. there is a single-valued function $v = v(x,\, y)$ such that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad , \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

and this v is a single-valued conjugate function of u.

Uniqueness If v_1 and v_2 are two single-valued conjugate functions of u, then $f_1 = u + iv_1$ and $f_2 = u + iv_2$ are both analytic on Ω, and $f_1 - f_2 = i\,(v_1 - v_2)$ is analytic on Ω, i.e. $f_1 - f_2$ is analytic from the open set Ω into the imaginary axis. By the open mapping theorem, $f_1 - f_2$ must be a constant, that is, there exists a constant $K \in \mathbb{R}$ such that $f_1 = f_2 + iK$. $\square$

6.2 The Mean-Value Property

In what follows, we will often use $(x,\, y) \in \mathbb{R}^2$ and $z = x + iy \in \mathbb{C}$ interchangeably, and allow ourselves this abuse of notation for the sake of the simplicity of the expressions.

Theorem 20. **(Mean-Value Theorem)** Let Ω be an open connected set in $\mathbb{C}$, $u : \Omega \to \mathbb{R}$ be a harmonic function on Ω, and $\overline{D}_R(z_0) \subset \Omega$ be a closed disk in Ω. Then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(z_0 + Re^{i\theta}\right)\,d\theta$$

Proof Since $\overline{D}_R(z_0) \subset \Omega$, there exists a (simply connected) open disk $D_{R'}(z_0)$ such that $\overline{D}_R(z_0) \subset D_{R'}(z_0) \subset \Omega$. By a Theorem 4.6.1, u has a harmonic conjugate v on $D_{R'}(z_0)$, and the function $f = u + iv$ is analytic on $D_{R'}(z_0)$. So, by the Cauchy Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + Re^{i\theta}\right)\,d\theta$$

Taking the real part of this equality, we get the Mean-Value Theorem for u. $\square$

Theorem 21. A nonconstant harmonic function has neither a maximum nor a minimum in any open connected set in which it is defined. Consequently, if a nonconstant harmonic function is defined on a closed bounded set E, its maximum and minimum are taken on the boundary of E.

Proof Suppose u reaches a maximum M at a point z_0 in the interior of Ω. There exists $R > 0$ such that $D_R(z_0) \subset \Omega$ and $\forall\, z \in D_R(z_0)$, $u(z) \leq u(z_0)$. Suppose there exists $a \in D_R(z_0)$ such that $u(a) < u(z_0) = M$. Consider the circle with radius r centered in z_0 and going through a, By the mean-value theorem,

$$M = u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) d\theta < M$$

This is a contradiction.

To obtain the result regarding the minimum, apply the same proof to the harmonic function $w = -u$. $\square$

Corollary If u_1 and u_2 are two continuous functions on a closed bounded set E which are harmonic in the interior of E and such that $u_1 = u_2$ on the boundary of E, then $u_1 = u_2$ in E.

In other words, functions satisfying the conditions above are uniquely determined by their values on the boundary.

6.3 Poisson's Formula

Theorem 22. (Poisson's Formula) Suppose that u is harmonic on $D_R(0)$ and continuous on $\overline{D}_R(0)$. Then the value of u at each point $a = re^{i\varphi} \in D_R(0)$ is given by

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z)\, d\theta = \frac{1}{2\pi} \int_{|z|=R} Re\left(\frac{z+a}{z-a}\right) u(z)\, d\theta \tag{30}$$

$$\iff u(re^{i\varphi}) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |r|^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} u(Re^{i\theta})\, d\theta$$

Proof For each $a \in D_R(0)$, and for each $|a| < \rho < R$, let $S : \overline{D}_1(0) \to \overline{D}_\rho(0)$ be the onto linear transformation defined by

$$z = S(\zeta) = \rho \cdot B_{-a/\rho}(\zeta) = \rho \cdot \frac{\zeta + (a/\rho)}{1 + (\overline{a}/\rho)\,\zeta} = \frac{\rho\,(\rho\,\zeta + a)}{\rho + \overline{a}\,\zeta}$$

such that $S(0) = a$. Since S is analytic in $\overline{D}_1(0)$ and u is harmonic on $\overline{D}_\rho(0)$, the function $\zeta \in D_1(0) \mapsto u\left(S(\zeta)\right)$ is harmonic on $\overline{D}_1(0)$. By the mean value property, we can write

$$u(a) = u\left(S(0)\right) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{u\left(S(\zeta)\right)}{\zeta - 0} d\zeta = -\frac{i}{2\pi} \int_{|\zeta|=1} u\left(S(\zeta)\right) \frac{d\zeta}{\zeta}$$

Since $z = S(\zeta) = \rho \cdot B_{-a/\rho}(\zeta) : \overline{D}_1(0) \to \overline{D}_\rho(0)$ is bijective with inverse $S^{-1}(z)$ given by

$$\zeta = S^{-1}(z) = \frac{1}{\rho} \cdot B_{a/\rho}(z/\rho) = \frac{\rho\,(z-a)}{\rho^2 - \overline{a}z} \implies \frac{d\zeta}{\zeta} = \left(\frac{1}{z-a} + \frac{\overline{a}}{\rho^2 - \overline{a}z}\right) dz$$

and since $|z| = |S(\zeta)| = \rho \iff |\zeta| = 1$, so by setting $z = \rho e^{i\theta}$, $dz = iz\, d\theta$ and $\rho^2 = z\overline{z}$ on $|z| = \rho$, we have

$$\frac{d\zeta}{\zeta} = \left(\frac{i\,z}{z-a} + \frac{i\,z\overline{a}}{\rho^2 - \overline{a}z}\right) d\theta = \left(\frac{i\,z}{z-a} + \frac{i\,\overline{a}}{\overline{z} - \overline{a}}\right) d\theta = i\left(\frac{\rho^2 - |a|^2}{|z-a|^2}\right) d\theta$$

so that

$$u(a) = \frac{1}{2\pi}\int_{|z|=\rho} \frac{\rho^2 - |a|^2}{|z-a|^2}\, u(z)\, d\theta \quad \forall\ |a| < \rho < R \implies u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}\, u(z)\, d\theta$$

by letting $\rho \to R$ and the uniform continuity of u on $\overline{D}_R(0)$. This formula is known as Poisson's formula. We may get an alternate form for it by observing that

$$\frac{R^2 - |a|^2}{|z-a|^2} = Re\left(\frac{R^2 - |a|^2}{|z-a|^2}\right) = Re\left(\frac{z}{z-a} + \frac{\overline{a}}{\overline{z}-\overline{a}}\right) = \frac{1}{2}\left(\frac{z+a}{z-a} + \frac{\overline{z}+\overline{a}}{\overline{z}-\overline{a}}\right) = Re\left(\frac{z+a}{z-a}\right)$$

We have shown that Poisson's formula could also be written as

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} Re\left(\frac{z+a}{z-a}\right) u(z)\, d\theta$$

which may yet again be rewritten as follows: for each $a \in D_R(0)$,

$$u(a) = Re\left(\frac{1}{2\pi i}\int_{|z|=R} \frac{z+a}{z-a}\,\frac{u(z)}{z}\, dz\right) \tag{31}$$

Since the function in parenthesis (31) is an analytic function of a for all $|a| < R$ (cf. 4.2.3 Lemma 3), the expression above implies that u is the real part of the analytic function

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{\zeta+z}{\zeta-z}\,\frac{u(\zeta)}{\zeta}\, d\zeta + iK \quad , \quad K \in \mathbb{R} \tag{32}$$

which is known as Schwarz's formula. $\square$

Remark One can prove that the Poisson's Formula holds for the closed disk $\overline{D}_R(0)$ as follows: For each $0 < \delta < 1$, since the function $\widetilde{u}$ be defined by $\widetilde{u}(z) := u(\delta z)$ is harmonic in $\overline{D}_R(0)$, so, for each $a \in D_R(0)$, we can write

$$u(\delta a) = \widetilde{u}(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}\, \widetilde{u}(z)\, d\theta = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}\, u(\delta z)\, d\theta$$

Now, u is continuous on the compact set $\overline{D}_R(0)$, so it is uniformly continuous on that set by the Heine-Cantor theorem. Taking the limit $\delta \to 1$ in the modified Poisson's formula above, we find by uniform continuity that Poisson's Formula holds for the closed disk $\overline{D}_R(0)$ as well.

6.4 Schwarz's Theorem

Poisson's formula can be viewed as a way to define a harmonic function inside a disk from the values $u(z)$ on the circle $|z| = R$ of a function u which may only be defined on that circle.

A natural question then is: does this function have boundary value $u(z)$ on $|z| = R$?

Schwarz's theorem, given below, answers this question.

Theorem 23. (Schwarz's Theorem) For each $z \in D_1(0)$ and $\theta \in \mathbb{R}$, let $K(\theta, z)$ be the Poisson kernel defined by

$$K(\theta, z) = Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)$$

Given a piecewise continuous function u on $[0,\, 2\pi]$, the Poisson integral

$$P_u(z) = \frac{1}{2\pi}\int_0^{2\pi} K(\theta, z)\, u(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) u(\theta)\, d\theta$$

is harmonic in $D_1(0)$ and $\lim_{z\to e^{i\theta_0}} P_u(z) = u(\theta_0)$ if u is continuous at θ_0, i.e. $P_u(z)$ is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta f(z) = 0 & \text{if } z \in D_1(0) \\ f(z) = u(\theta) & \text{if } z = e^{i\theta} \in \partial D_1(0) \end{cases}$$

Remark Note that if $z = re^{i\varphi} \in D_1(0)$, since

$$\frac{e^{i\theta}+z}{e^{i\theta}-z} = \frac{1+re^{-i(\theta-\varphi)}}{1-re^{-i(\theta-\varphi)}} = \left(1+re^{-i(\theta-\varphi)}\right)\sum_{n=0}^{\infty} r^n e^{-in(\theta-\varphi)} = 1+2\sum_{n=1}^{\infty} r^n e^{-in(\theta-\varphi)}$$

$$\Longrightarrow K(\theta,z) = Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = 1+2\,Re\sum_{n=1}^{\infty} r^n e^{-in(\theta-\varphi)} = \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}$$

$$\Longrightarrow K(\theta,z) > 0 \ \forall\,\theta\in\mathbb{R}\ ,\ \frac{1}{2\pi}\int_0^{2\pi} K(\theta,z)\,d\theta = 1,\ \text{ and } \forall\,\delta>0 \text{ such that}$$

$$0<\delta\le|\theta-\varphi|\le\pi,\ \text{ since } 1-2r\cos(\theta-\varphi)+r^2 \ge r^2\sin^2\delta > 0,$$

$$\lim_{r\to 1^-} K(\theta, re^{i\varphi}) = 0 \text{ uniformly in } \theta \text{ for all } 0<\delta\le|\theta-\varphi|\le\pi$$

Remark Let $\mathrm{PC}([0,2\pi]) = \{u : [0,2\pi]\to\mathbb{R} \mid u \text{ is piecewise continuous on } [0,2\pi]\}$. Then

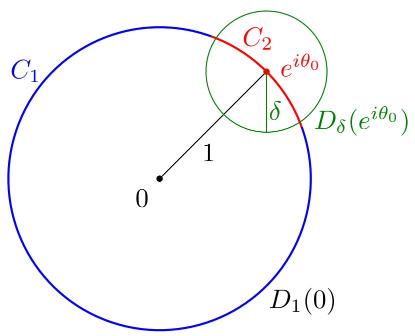
- P_u is harmonic in $D_1(0)$ for each $u\in\mathrm{PC}([0,2\pi])$, since it is the real part of an analytic function defined by Schwarz's formula in the proof of Poisson Formula (Theorem 22).
- P is a linear operator from $\mathrm{PC}([0,2\pi])$ to harmonic functions P_u on $D_1(0)$ since $P_{u_1+u_2} = P_{u_1}+P_{u_2}$ and $P_{cu} = cP_u$ for any $u_1,\,u_2\in\mathrm{PC}([0,2\pi])$ and constant $c\in\mathbb{R}$.

Since $P_1 = 1,\ P_c = c$ for all $c\in\mathbb{R}$, and $P_u\ge 0$ for all $u\in\mathrm{PC}([0,2\pi])$ such that $u(\theta)\ge 0$ for all $\theta\in[0,\,2\pi]$, we have $m\le P_u\le M$ if $u\in\mathrm{PC}([0,2\pi])$ such that $m\le u(\theta)\le M$ for all $\theta\in[0,\,2\pi]$.

Proof Without loss of generality, assume $u(\theta_0)=0$ (otherwise, consider $u-u(\theta_0)$ and note that $P_{u-u(\theta_0)} = P_u - P_{u(\theta_0)} = P_u - u(\theta_0)$).

- $\forall\,\varepsilon>0$, since u is continuous at θ_0, there exists an open arc C_2 in $\partial D_1(0)$ such that $e^{i\theta_0}\in C_2$ and $|u(\theta)|<\varepsilon/2$ for all $e^{i\theta}\in C_2$.
- Let $C_1 = \partial D_1(0)\setminus C_2$ be the complement C_1 of C_2 in $\partial D_1(0)$, and $u_1,\,u_2$ be defined by

$$u_1(\theta) = \begin{cases} u(\theta) & \text{for } \theta \text{ such that } e^{i\theta}\in C_1 \\ 0 & \text{otherwise}\end{cases} \qquad u_2(\theta) = \begin{cases} u(\theta) & \text{for } \theta \text{ such that } e^{i\theta}\in C_2 \\ 0 & \text{otherwise}\end{cases}$$



Note that

– $P_u = P_{u_1} + P_{u_2}$ by the linearity of P, where P_{u_1} and P_{u_2} are harmonic everywhere except on C_1 and C_2, respectively, since they are line integrals over C_1 or C_2 (cf. 4.2.3 Lemma 3).

– $|P_{u_2}(z)| < \frac{\varepsilon}{2}$ for all $z \in D_1(0)$ since $|u_2(\theta)| < \frac{\varepsilon}{2}$ for all $\theta \in [0, 2\pi]$ and $\frac{1}{2\pi}\int_0^{2\pi} K(\theta, z)\, d\theta = 1$.

– for each $\varepsilon > 0$, since P_{u_1} is harmonic everywhere except on C_1, P_{u_1} is continuous at $e^{i\theta_0} \in C_2$, and P_{u_1} is zero on C_2 by the remark above, so $\lim_{|z|\le 1,\, z\to e^{i\theta_0}} P_{u_1}(z) = u(\theta_0) = 0$ and there exists a $\delta > 0$ such that if $z \in \overline{D}_1(0)$ and $|z - e^{i\theta_0}| < \delta$, then $|P_{u_1}(z)| < \frac{\varepsilon}{2}$.

We conclude that if $z \in D_1(0)$ such that $|z - e^{i\theta_0}| < \delta$, then

$$|P_u(z)| \le |P_{u_1}(z)| + |P_{u_2}(z)| < \varepsilon$$

Since ε is arbitrary, this completes our proof. □

6.5 The Reflection Principle

The idea of the Schwarz reflection principle is to extend an analytic function $f : \Omega \to \mathbb{C}$ to a larger domain, with the ultimate goal to find the maximal domain on which f can be defined and analytic.

Recall that if $f(z) = u(z) + iv(z)$ is analytic on Ω, then $\overline{f(\overline{z})} = u(\overline{z}) - iv(\overline{z})$ is analytic on $\widetilde{\Omega} = \{z \in \mathbb{C} \mid \overline{z} \in \Omega\}$ by the definition.

Now, if f is an analytic function defined on an open connected set Ω which is symmetric about the x-axis, and $f(z) = \overline{f(\overline{z})}$, then f is real on the intersection of the x-axis with Ω. We have the following converse:

Theorem 24. Let Ω be an open connected set which is symmetric with respect to the x-axis, and let $\Omega^+ = \Omega \cap \{Im\,(z) > 0\}$ and $\sigma = \Omega \cap \{Im\,(z) = 0\}$. If f is continuous on $\Omega^+ \cup \sigma$, analytic on Ω^+, and real for all $z \in \sigma$, then f has an analytic continuation to all of Ω such that $f(z) = \overline{f(\overline{z})}$.

The theorem above follows from the following theorem regarding harmonic functions, which we will prove first:

Theorem 24′. Suppose v is continuous on $\Omega^+ \cup \sigma$, harmonic on Ω^+, and zero on σ. Then v has a harmonic extension to Ω satisfying $v(z) = -v(\overline{z})$.

Proof of Theorem 24′ Let $\Omega^- = \Omega \cap \{Im\,(z) < 0\}$, and V be an extension on Ω defined by

$$V(z) = \begin{cases} v(z) & \text{if } z \in \Omega^+, \\ 0 & \text{if } z \in \sigma, \\ -v(\overline{z}) & \text{if } z \in \Omega^-. \end{cases}$$

To show that V is harmonic in Ω, it suffices to show that V is harmonic on σ since $v(z) = \operatorname{Im} f(z)$ and $-v(\overline{z}) = \operatorname{Im} \overline{f(\overline{z})}$ are harmonic in Ω^+ and Ω^-, respectively.

For each $z_0 \in \sigma$, let $\delta > 0$ be small enough such that $\overline{D}_\delta(z_0) \subset \Omega$, and P_V be the Poisson integral of V with respect to $\partial D_\delta(z_0)$. Then P_V is harmonic in $D_\delta(z_0)$ and continuous on $\overline{D}_\delta(z_0)$ by Schwarz's Theorem (Theorem 23). Since

- $V - P_V$ is harmonic in the upper half disk $\overline{D}_\delta(z_0) \cap \left(\Omega^+ \cup \sigma\right)$,
- $V - P_V = 0$ on the upper semi-circle $C_\delta(z_0) \cap \left(\Omega^+ \cup \sigma\right)$ by Schwarz's Theorem,

- $V(z) = 0$ on σ by construction, and if $z \in \sigma$,

$$P_V(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{\delta\, e^{i\theta}+z}{\delta\, e^{i\theta}-z}\right) V(\delta\, e^{i\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta^2-|z|^2}{|\delta\, e^{i\theta}-z|^2} V(\delta\, e^{i\theta})\, d\theta = 0,$$

  where the last equality follows from the symmetry of the integrand,

we conclude that $V = P_V$ in $\overline{D}_\delta(z_0) \cap (\Omega^+ \cup \sigma)$ by the maximum principle. Similarly, we can show that $V = P_V$ in $\overline{D}_\delta(z_0) \cap (\Omega^- \cup \sigma)$. Hence, $V = P_V$ in $D_\delta(z_0)$, and V as constructed above is harmonic in z_0. Since z_0 is an arbitrary point in σ, V is harmonic on σ.

Applying the maximum principle to V on overlapping disks, v can be extended to a harmonic function V defined on all of Ω. $\square$

Proof of Theorem 24 Consider $f = u + iv$ defined on Ω^+, We want to verify that the extension of f defined by $f(z) = \overline{f(\overline{z})} = u(\overline{z}) - iv(\overline{z})$ is an analytic extension of f on all of Ω.

Let $z_0 \in \sigma$, $\overline{D}_\delta(z_0) \subset \Omega$, and V be a harmonic extension of v to $D_\delta(z_0)$ as before. Since $-v$ has a conjugate harmonic function u on $D_\delta(z_0) \cap \Omega^-$, let U be the harmonic conjugate of V defined by

$$U(z) = \begin{cases} u(z) & \text{if } z \in \Omega^+ \cap D_\delta(z_0), \\ 0 & \text{if } z \in \sigma \quad \cap D_\delta(z_0), \\ u(\overline{z}) & \text{if } z \in \Omega^- \cap D_\delta(z_0). \end{cases}$$

Consider the function $g(z) := U(z) - U(\overline{z})$ on $D_\delta(z_0)$. Note that

- since g is harmonic on $D_\delta(z_0)$, the function $h(z) := \dfrac{\partial g}{\partial x} - i\dfrac{\partial g}{\partial y}$ is analytic on $D_\delta(z_0)$.
- since $g(z) = 0$ for all $z \in \sigma \cap D_\delta(z_0)$, $\dfrac{\partial g}{\partial x}(z) = 0$ for all $z \in \sigma \cap D_\delta(z_0)$.
- $\dfrac{\partial g}{\partial y}(z) = 2\dfrac{\partial U}{\partial y}(z) = -2\dfrac{\partial V}{\partial x}(z) = 0$ for all $z \in \sigma \cap D_\delta(z_0)$.

Thus, the analytic function $h(z) := \dfrac{\partial g}{\partial x} - i\dfrac{\partial g}{\partial y} = 0$ for all $z \in \sigma \cap D_\delta(z_0)$. So, by the Uniqueness Theorem, $h \equiv 0 \iff \dfrac{\partial g}{\partial x} \equiv 0 \equiv \dfrac{\partial g}{\partial y}$ in $D_\delta(z_0)$, that is, $g(z) = 0 \iff U(z) = U(\overline{z})$ for all $z \in D_\delta(z_0)$ which implies that $f(z) = U(z) + iV(z)$ is an appropriate analytic continuation of f on all of $D_\delta(z_0)$ such that $f(z) = \overline{f(\overline{z})}$.

By applying the maximum principle to U on overlapping disks, U and hence f can be extended on all of Ω. $\square$

4.5.3 Evaluation of Definite Integrals

Recall that if f is analytic in a simply connected domain D except for isolated singularities at $\{\alpha_k\}_{k=1}^m$, and if γ is a closed curve not intersecting any of the singularities, then

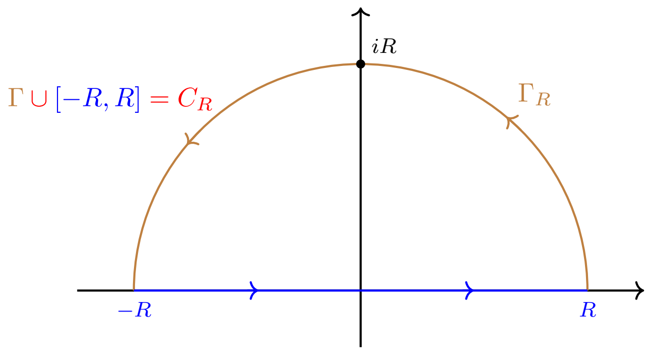
$$\int_\gamma f = 2\pi i \sum_{k=1}^m n(\gamma, \alpha_k)\, \text{Res}\,(f; \alpha_k),$$

Type (1) Integrals of the Form $\int_{-\infty}^{\infty} (P(x)/Q(x))\,dx$, where P, Q are polynomials such that $Q(x) \neq 0$ for $x \in \mathbb{R}$, and $\deg Q \geq \deg P + 2$. By the Residue Theorem,

$$\lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)}\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}\left(\frac{P(z)}{Q(z)}; \alpha_k\right),$$

where $\{\alpha_k\}_{k=1}^{m}$ are singularities of $\dfrac{P(z)}{Q(z)}$ in $\{z \mid \operatorname{Im} z > 0\}$, $C_R = \Gamma_R \cup [-R, R]$, and

- $\Gamma_R = \{|z| = R,\ \operatorname{Im} z \geq 0\} = \{Re^{it} \mid 0 \leq t \leq \pi\}$,
- $[-R, R]$ is the line segment from $z = -R$ to $z = R$.



This implies that $\lim_{R\to\infty} \left| \int_{\Gamma_R} \frac{P(z)}{Q(z)}\,dz \right| = 0$, and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = \lim_{R\to\infty} \left[\int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)}\,dz \right] = \lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)}\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}\left(\frac{P(z)}{Q(z)}; \alpha_k\right).$$

where $\{\alpha_k\}_{k=1}^{m}$ are singularities of $P(z)/Q(z)$ in $\{z \mid \operatorname{Im} z > 0\}$.

Example Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$.

Solution Since $\dfrac{1}{z^4+1}$ has simple poles at $\alpha_k = e^{i(\pi/4+(k-1)\pi/2)}$ inside C_R for each $R > 1$, $k = 1, 2$, with

$$\operatorname{Res}\left(\frac{1}{z^4+1}; \alpha_k\right) = \frac{1}{4\alpha_k^3} = -\frac{\alpha_k}{4} \quad \text{for } k = 1, 2,$$

so

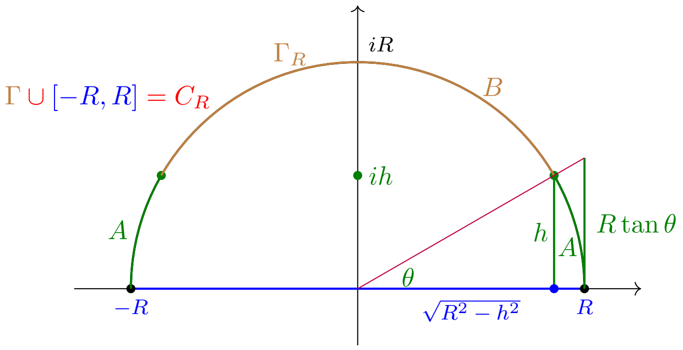
$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^4+1}\,dx = \lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1}\,dz = 2\pi i \sum_{k=1}^{2} \operatorname{Res}\left(\frac{1}{z^4+1}; \alpha_k\right) = \frac{\pi\sqrt{2}}{2}.$$

Example Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$.

Type (2) Integrals of the Form $\int_{-\infty}^{\infty} (P(x)/Q(x)) \cos x\,dx$ or $\int_{-\infty}^{\infty} (P(x)/Q(x)) \sin x\,dx$, where P, Q are polynomials such that $Q(x) \neq 0$ for real x (except perhaps at a zero of $\cos x$ or $\sin x$), and $\deg Q \geq \deg P + 1$. Applying the Residue Theorem to $(P(z)/Q(z))\, e^{iz}$ around C_R

$$\lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)}\, e^{iz}\, dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}\left(\frac{P(z)}{Q(z)}\, e^{iz}; \alpha_k\right),$$

where $\{\alpha_k\}_{k=1}^m$ are singularities of $P(z)/Q(z)$ in $\{z \mid \operatorname{Im} z > 0\}$, $C_R = \Gamma_R \cup [-R, R]$ as above.



Since $\deg Q \geq \deg P + 1$, there exists K, $M > 0$ such that $\left|\frac{P(z)}{Q(z)}\right| \leq \frac{K}{R}$ for all $|z| = R \geq M$. Let $h \leq \frac{\sqrt{3}}{2}R \iff \frac{h^2}{R^2} \leq \frac{3}{4} \iff 1 - \frac{h^2}{R^2} \geq \frac{1}{4}$ and $\Gamma_R = A \cup B$, where

- $A = \{|z| = R,\, 0 \leq \operatorname{Im} z \leq h\} \implies \int_A |dz| = 2R\,\theta \leq 2R\tan\theta = 2R \cdot \frac{h}{\sqrt{R^2-h^2}} \leq \frac{2h}{1/2} = 4h,$
- $B = \{|z| = R,\, \operatorname{Im} z \geq h\} \implies \int_B |e^{iz}|\,|dz| \leq e^{-h}\,\pi R.$

Now $|e^{iz}| = |e^{ix-y}| = e^{-y}$, we have $|e^{iz}| \leq 1$ for $z \in A$, $|e^{iz}| \leq e^{-h}$ for $z \in B$, and

$$\left|\int_A \frac{P(z)}{Q(z)}\, e^{iz}\, dz\right| \leq \frac{K}{R} \cdot 4h \text{ for } h \leq \frac{\sqrt{3}}{2}R \quad \text{and} \quad \left|\int_B \frac{P(z)}{Q(z)}\, e^{iz}\, dz\right| \leq \frac{K}{R} \cdot e^{-h} \cdot \pi\, R,$$

so by setting $h = \sqrt{R}/2$ and letting $R \to \infty$, we have $\lim_{R\to\infty} \left|\int_{\Gamma_R} \frac{P(z)}{Q(z)}\, e^{iz}\, dz\right| = 0$, and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, e^{ix}\, dx = \lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)}\, e^{iz}\, dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}\left(\frac{P(z)}{Q(z)}\, e^{iz}; \alpha_k\right),$$

where $\{\alpha_k\}_{k=1}^m$ are singularities of $P(z)/Q(z)$ in $\{z \mid \operatorname{Im} z > 0\}$.

Example Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$.

Solution Since

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = \sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(ix)^{2k+1}}{(2k+1)!} = \cos x + i \sin x,$$

and $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we have $\frac{\sin x}{x} = \operatorname{Im}\left(\frac{e^{ix}-1}{x}\right)$ and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}\, dx.$$

Since $\dfrac{e^{iz}-1}{z}$ is analytic for all $z \neq 0$ and has a removable singularity at $z=0$, so by Cauchy's Theorem,

$$0 = \int_{C_R} \frac{e^{iz}-1}{z}\,dz = \int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx + \int_{\Gamma_R} \frac{e^{iz}-1}{z}\,dz \quad \text{for all } R>0.$$

Thus

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx = \lim_{R\to\infty} \int_{\Gamma_R} \frac{1-e^{iz}}{z}\,dz = \lim_{R\to\infty} \int_{\Gamma_R} \frac{1}{z}\,dz - \lim_{R\to\infty} \int_{A\cup B} \frac{e^{iz}}{z}\,dz = \pi i,$$

where $A = \{|z|=R,\ \operatorname{Im} z \leq \sqrt{R}\}$ and $B = \{|z|=R,\ \operatorname{Im} z \geq \sqrt{R}\}$, and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}\,dx = \operatorname{Im}\left(\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx\right) = \operatorname{Im}\left(\lim_{R\to\infty} \int_{\Gamma_R} \frac{1-e^{iz}}{z}\,dz\right) = \pi.$$

Definition Let $f:\mathbb{R}\to\mathbb{R}$ or $f:\mathbb{R}\to\mathbb{C}$. The Fourier transform of f is given by

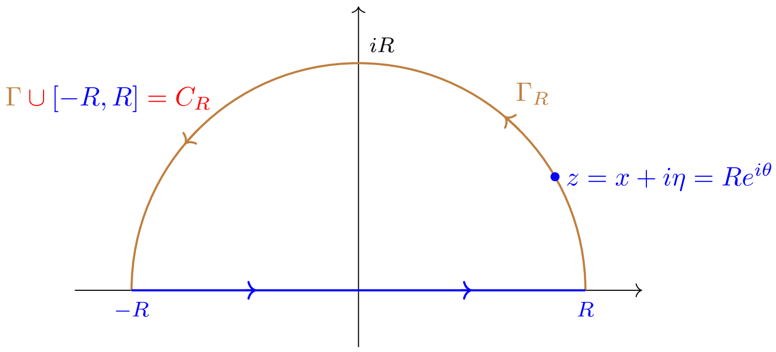
$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y}\,dx.$$

The inversion formula is

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(y) e^{2\pi i y x}\,dy.$$

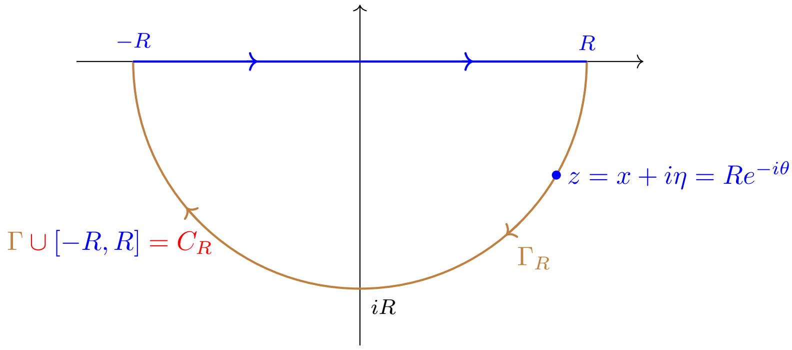
Example Let $f(x) = \dfrac{1}{1+x^2}$ for $x \in \mathbb{R}$. Find $\widehat{f}(y) = \displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-2\pi i x y}\,dx$.

For $y \leq 0$ and $z = x + i\eta \in \{|z| = R \mid \eta > 0\}$, since $|e^{-2\pi i y z}| = |e^{-2\pi i y(x+i\eta)}| = e^{2\pi y \eta} \leq e^0 = 1$, $\displaystyle\lim_{R\to\infty} \int_{\Gamma_R} \frac{|e^{-2\pi i y z}|\,|dz|}{|1+z^2|} \leq \lim_{R\to\infty} \int_{\Gamma_R} \frac{|dz|}{|z|^2-1} = \lim_{R\to\infty} \frac{\pi R}{R^2-1} = 0$, and since $n(C_R, i) = 1$, we have



$$\widehat{f}(y) = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{-2\pi i y x}}{1+x^2}\,dx = \lim_{R\to\infty} \int_{C_R} \frac{e^{-2\pi i y z}}{1+z^2}\,dz = 2\pi i \operatorname{Res}\left(\frac{e^{-2\pi i y z}}{1+z^2}; i\right) = 2\pi i \cdot \frac{e^{2\pi y}}{2i} = \pi e^{2\pi y}.$$

For $y \geq 0$ and $z = x + i\eta \in \{|z| = R \mid \eta < 0\}$, since $|e^{-2\pi i y z}| = |e^{-2\pi i y(x+i\eta)}| = e^{2\pi y \eta} \leq e^0 = 1$, $\displaystyle\lim_{R\to\infty} \int_{\Gamma_R} \frac{|e^{-2\pi i y z}|\,|dz|}{|1+z^2|} \leq \lim_{R\to\infty} \int_{\Gamma_R} \frac{|dz|}{|z|^2-1} = \lim_{R\to\infty} \frac{\pi R}{R^2-1} = 0$, and since $n(C_R, i) = 1$, we have

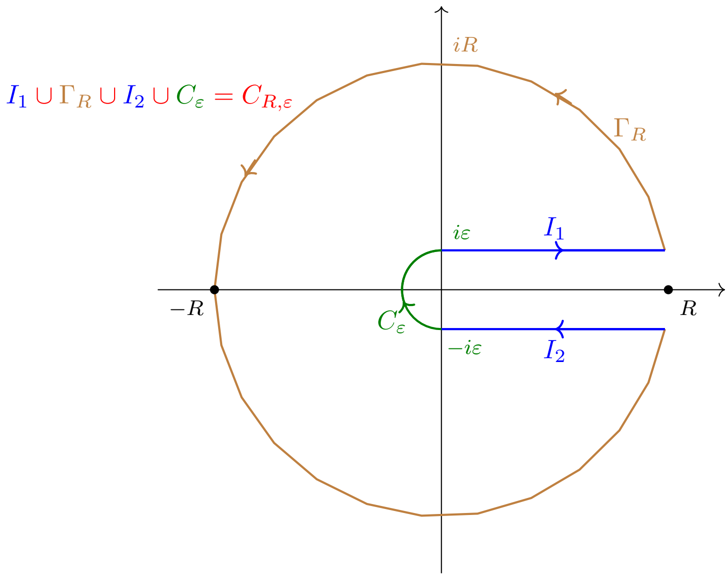


$$\widehat{f}(y) = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{-2\pi i y x}}{1+x^2}\, dx = \lim_{R\to\infty} \int_{C_R} \frac{e^{-2\pi i y z}}{1+z^2}\, dz = 2\pi i \cdot n(C_R, -i)\, \mathrm{Res}\left(\frac{e^{-2\pi i y z}}{1+z^2}; -i\right)$$
$$= -2\pi i \cdot \frac{e^{-2\pi y}}{-2i} = \pi e^{-2\pi y}.$$

Hence we have

$$\widehat{f}(y) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-2\pi i x y}\, dx = \begin{cases} \pi e^{2\pi y} & \text{if } y \le 0 \\ \pi e^{-2\pi y} & \text{if } y \ge 0 \end{cases} = \pi e^{-2\pi |y|} \quad \forall\, y \in \mathbb{R}.$$

Type $(3A)$ Integral of the Form $\displaystyle\int_0^{\infty} (P(x)/Q(x))\, dx$, where P, Q be polynomials such that $\deg Q \ge \deg P + 2$ and $Q(x) \ne 0$ for $x \ge 0$. Applying the Residue Theorem to $(P(z)/Q(z)) \log z$ around $C_{R,\varepsilon}$, we have



$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{C_{R,\varepsilon}} \frac{P(z)}{Q(z)} \log z\, dz = 2\pi i \sum_{k=1}^{m} \mathrm{Res}\left(\frac{P(z)}{Q(z)} \log z; \alpha_k\right),$$

where $\{\alpha_k\}_{k=1}^m$ are singularities of $P(z)/Q(z)$, $C_{R,\varepsilon} = I_1 \cup \Gamma_R \cup I_2 \cup C_\varepsilon$ such that

- I_1 is the line segment from $z = i\varepsilon$ to $z = \sqrt{R^2 - \varepsilon^2} + i\varepsilon$,

- Γ_R is the circular arc of radius R from $z = \sqrt{R^2 - \varepsilon^2} + i\varepsilon$ to $z = \sqrt{R^2 - \varepsilon^2} - i\varepsilon$,
- I_2 is the line segment from $z = \sqrt{R^2 - \varepsilon^2} - i\varepsilon$ to $z = -i\varepsilon$,
- C_ε is the circular arc of radius ε from $z = -i\varepsilon$ to $z = i\varepsilon$.

Note that for $R > 1 > \varepsilon > 0$, the inside of $C_{R,\varepsilon}$ is a simply connected domain not containing 0 and $\log z = \log|z| + i\operatorname{Arg} z$ is analytic in $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \le 0\}$ if $\operatorname{Arg} z$ is defined such that $0 < \operatorname{Arg} z < 2\pi$.

Since $P(z)/Q(z)$ is continuous at 0, $|\log z| < \log|z| + 2\pi$ for $z \in C_\varepsilon$, and since $(|P(z)|/|Q(z)|) \le B/|z|^2$ for some constant B and for $z \in \Gamma_R$,

$$\begin{aligned}
&\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\Gamma_R \cup C_\varepsilon} \frac{P(z)}{Q(z)} \log z\, dz = 0,\\
&\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{I_1} \frac{P(z)}{Q(z)} \log z\, dz = \int_0^\infty \frac{P(x)}{Q(x)} \log x\, dx,\\
&\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{I_2} \frac{P(z)}{Q(z)} \log z\, dz = -\int_0^\infty \frac{P(x)}{Q(x)} \big(\log x + 2\pi i\big)\, dx,
\end{aligned}$$

$$\Longrightarrow \int_0^\infty \frac{P(x)}{Q(x)}\, dx = -\sum_{k=1}^m \operatorname{Res}\left(\frac{P(z)}{Q(z)} \log z; \alpha_k\right), \text{ where } \{\alpha_k\}_{k=1}^m \text{ are singularities of } P(z)/Q(z).$$

Example Since $\alpha_k = e^{i(\pi/3 + 2(k-1)\pi/3)}$ is a simple pole of $\dfrac{\log z}{z^3+1}$, so

$$\operatorname{Res}\left(\frac{\log z}{z^3+1}; \alpha_k\right) = \frac{\log \alpha_k}{3\alpha_k^2} = -\frac{i[\pi/3 + 2(k-1)\pi/3]\alpha_k}{3} \quad \text{for } k = 1, 2, 3,$$

and

$$\int_0^\infty \frac{1}{x^3+1}\, dx = -\sum_{k=1}^3 \operatorname{Res}\left(\frac{\log z}{z^3+1}; \alpha_k\right) = \sum_{k=1}^3 \frac{i[\pi/3 + 2(k-1)\pi/3]\alpha_k}{3} = \frac{2\pi\sqrt{3}}{9}.$$

Type $(3B)$ Integrals of the Form $\displaystyle\int_a^\infty \frac{P(x)}{Q(x)}\, dx$, where P, Q are polynomials such that $\deg Q \ge \deg P + 2$ and $Q(x) \ne 0$ for $x \ge a$, can be evaluated in a similar manner by considering

$$\int_{C_{R,\varepsilon}} \frac{P(z)}{Q(z)} \log(z-a)\, dz \Longrightarrow \int_a^\infty \frac{P(x)}{Q(x)}\, dx = -\sum_{k=1}^m \operatorname{Res}\left(\frac{P(z)}{Q(z)} \log(z-a); \alpha_k\right),$$

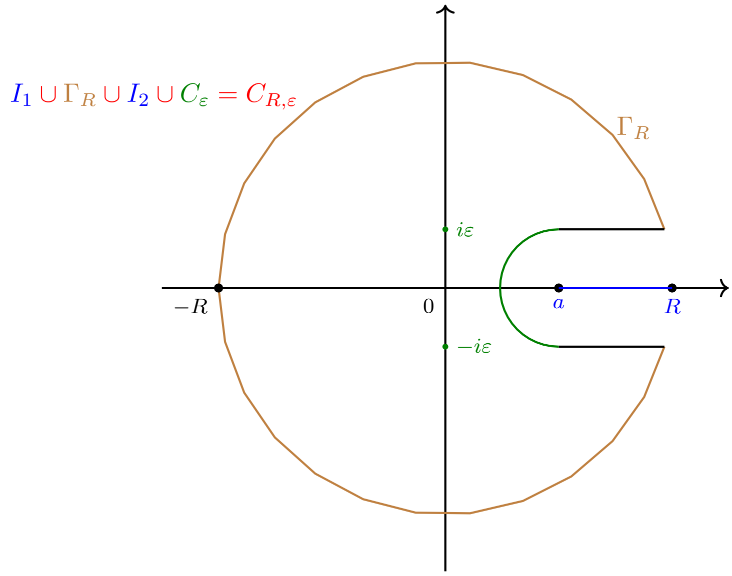
where $C_{R,\varepsilon}$ is a contour as follows and $\{\alpha_k\}_{k=1}^m$ are singularities of $P(z)/Q(z)$.

Type $(3C)$ Integrals of the Form $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{Q(x)}\, dx$, where $0 < \alpha < 1$, Q is a polynomial such that $\deg Q \ge 1$ and $Q(x) \ne 0$ for $x \ge 0$, can also be evaluated by considering

$$\int_{C_{R,\varepsilon}} \frac{z^{\alpha-1}}{Q(z)}\, dz,$$

where $C_{R,\varepsilon}$ is the "keyhole" contour as in $(3A)$. Since

$$z^{\alpha-1} = \begin{cases} e^{(\alpha-1)\log x} = x^{\alpha-1} & \text{along } I_1,\\ e^{(\alpha-1)\left(\log x + 2\pi i\right)} = x^{\alpha-1} e^{2\pi i(\alpha-1)} & \text{along } I_2,\end{cases}$$



we have

$$\left[1 - e^{2\pi i(\alpha-1)}\right] \int_0^\infty \frac{x^{\alpha-1}}{Q(x)}\,dx = 2\pi i \sum_{k=1}^m \operatorname{Res}\left(\frac{z^{\alpha-1}}{Q(z)}; \alpha_k\right), \quad \text{where } \{\alpha_k\}_{k=1}^m \text{ are zeros of } Q(z).$$

Example Since $z = -1$ is the only simple pole of $z^{-1+1/2}/\left(1+z\right)$ with $\operatorname{Res}\left(z^{-1+1/2}/\left(1+z\right); -1\right) = -i$, so

$$\left[1 - e^{2\pi i(-1+1/2)}\right] \int_0^\infty \frac{x^{-1+1/2}}{1+x}\,dx = 2\pi i \operatorname{Res}\left(\frac{z^{-1+1/2}}{1+z}; -1\right) = 2\pi \implies \int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \pi.$$

Type (4) Integrals of the Form $\displaystyle\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta$, where R is a rational function defined on the unit circle $|z| = 1$. By setting $z = e^{i\theta}$ for $0 \le \theta \le 2\pi$, we have

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\implies \int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta = \int_{|z|=1} R\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)\frac{dz}{iz}$$

and note that the last contour integral, as always, can be evaluated by the Residue Theorem.

Example

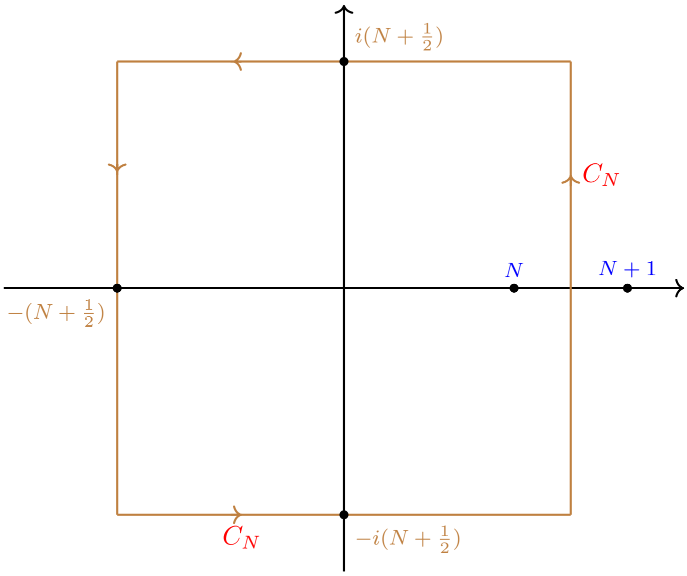
$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+4z+1} = 4\pi \operatorname{Res}\left(\frac{1}{z^2+4z+1}; \sqrt{3}-2\right) = \frac{2}{3}\pi\sqrt{3}.$$

Type (5) Sums of the Form $\displaystyle\sum_{-\infty}^{\infty} f(n)$, where $|f(z)| \le A/|z|^2$ so that $\displaystyle\lim_{z\to\infty} zf(z) = 0$. Since $\cot\pi z = \cos\pi z/\sin\pi z$ has a simple pole at each integer $z = n$ with

$$\operatorname{Res}\left(\frac{\pi\cos\pi z}{\sin\pi z}; n\right) = \frac{\pi\cos n\pi}{\pi\cos n\pi} = 1,$$

and by applying the Residue Theorem to the integral

$$\int_{C_N} f(z) \cdot \pi \cot \pi z \, dz,$$



where C_N is a simple closed contour enclosing the integers $n = 0, \pm 1, \pm 2, \ldots, \pm N$ and the poles of f which we assume to be finite in number, we have

$$\begin{aligned}
\int_{C_N} \pi f(z) \cot \pi z \, dz &= 2\pi i \left[\sum_{n=-N,\, n\neq \alpha_k}^{N} \text{Res}\,(\pi f(z) \cot \pi z; n) + \sum_{k=1}^{m} \text{Res}\,(\pi f(z) \cot \pi z; \alpha_k) \right] \\
&= 2\pi i \left[\sum_{n=-N,\, n\neq \alpha_k}^{N} f(n) + \sum_{k=1}^{m} \text{Res}\,(\pi f(z) \cot \pi z; \alpha_k) \right],
\end{aligned}$$

where $\{\alpha_k\}_{k=1}^{m}$ are poles of f. Note that

$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z} = i\,\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = i\frac{e^{i2\pi z} + 1}{e^{i2\pi z} - 1} = i\frac{e^{i2\pi x - 2\pi y} + 1}{e^{i2\pi x - 2\pi y} - 1},$$

and

- if $z = \pm\left(N + \frac{1}{2}\right) + iy \in C_N$, then

$$|\cot \pi z| = \left| \frac{e^{\pm \pi i - 2\pi y} + 1}{e^{\pm \pi i - 2\pi y} - 1} \right| = \left| \frac{1 - e^{-2\pi y}}{1 + e^{-2\pi y}} \right| < \frac{1 + e^{-2\pi y}}{1 + e^{-2\pi y}} = 1,$$

- if $z = x \pm \left(N + \frac{1}{2}\right)i \in C_N$, then

$$|\cot \pi z| \leq \left| \frac{1 + e^{\mp \pi (2N+1)}}{1 - e^{\mp \pi (2N+1)}} \right| = \left| \frac{1 + e^{-\pi (2N+1)}}{1 - e^{-\pi (2N+1)}} \right| \leq \left| \frac{1 + e^{-\pi}}{1 - e^{-\pi}} \right| < 2.$$

This implies that there exists a constant A such that

$$\left|\int_{C_N} \pi\, f(z)\, \cot \pi z\, dz\right| \le A \max_{z\in C_N} |zf(z)|.$$

Since $\lim_{z\to\infty} zf(z) = 0$, we have

$$0 = \lim_{N\to\infty} \int_{C_N} \pi\, f(z)\, \cot \pi z\, dz = 2\pi i \left[\sum_{n=-\infty,\, n\neq\alpha_k}^{\infty} f(n) + \sum_{k=1}^{m} \text{Res}\, (\pi f(z) \cot \pi z; \alpha_k)\right],$$

and hence

$$(*) \qquad \sum_{n=-\infty,\, n\neq\alpha_k}^{\infty} f(n) = -\sum_{k=1}^{m} \text{Res}\, (\pi f(z) \cot \pi z; \alpha_k)\,, \quad \text{where } \{\alpha_k\}_{k=1}^m \text{ are poles of } f.$$

Example Since

$$\text{Res}\left(\frac{\pi \cot \pi z}{z^2}; n\right) = \frac{1}{n^2} \quad \forall\, n \in \mathbb{Z},\ n \neq 0,$$

by setting $f(z) = \dfrac{1}{z^2}$ in $(*)$, we have

$$\sum_{1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n=-\infty,\, n\neq 0}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\text{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right).$$

Since $\cot(-z) = \cot z$ which implies that $c_{2k} = 0$ for all k in the Laurent expansion of $\cot z$ at $z = 0$, and $c_{-1} = \lim_{z\to 0} z \cot z$, $c_1 = \lim_{z\to 0}\left(\cot z - \dfrac{c_{-1}}{z}\right)\dfrac{1}{z}$, $c_3 = \lim_{z\to 0}\left(\cot z - \dfrac{c_{-1}}{z} - c_1 z\right)\dfrac{1}{z^3}, \dots,$ etc., that is

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots \implies \frac{\pi \cot \pi z}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} - \frac{\pi^4 z}{45} + \cdots,$$

we have

$$\sum_{1}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\text{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right) = \frac{\pi^2}{6}.$$

Example Since

$$\text{Res}\left(\frac{\pi}{\sin \pi z}; n\right) = \frac{1}{\cos \pi n} = (-1)^n \quad \forall\, n \in \mathbb{Z},$$

and since

$$\csc^2 \pi z = 1 + \cot^2 \pi z \implies \csc \pi z \text{ is bounded on } C_N,$$

Thus we may conclude that

$$\lim_{N\to\infty} \int_{C_N} \pi\, f(z)\, \csc \pi z\, dz = 0,$$

and by the Residue Theorem, that

$$\sum_{n=-\infty,\, n\neq\alpha_k}^{\infty} (-1)^n\, f(n) = -\sum_{k=1}^{m} \text{Res}\, (\pi f(z) \csc \pi z; \alpha_k)\,,$$

where $\{\alpha_k\}_{k=1}^m$ are poles of f.

So, by setting $f(z) = \dfrac{1}{z^2}$, we have

$$\sum_{1}^{\infty} \frac{(-1)^{n-1}}{n^2} = -\frac{1}{2} \sum_{n=-\infty,\, n\neq 0}^{\infty} \frac{(-1)^n}{n^2} = -\frac{1}{2}\mathrm{Res}\left(\frac{\pi \csc \pi z}{z^2}; 0\right) = \frac{\pi^2}{12}$$

since

$$\frac{\pi \csc \pi z}{z^2} = \frac{1}{z^3} + \frac{\pi^2}{6z} + \frac{7\pi^4 z}{360} + \cdots .$$

Example For each $n \geq 0$, since

$$\binom{2n}{n} = \frac{1}{2\pi i}\int_C \frac{(1+z)^{2n}}{z^{n+1}}\,dz \implies \binom{2n}{n}\frac{1}{5^n} = \frac{1}{2\pi i}\int_C \frac{(1+z)^{2n}}{(5z)^n}\frac{dz}{z},$$

where $C = \{z \mid |z| = 1\}$, and since $\dfrac{|1+z|^2}{5|z|} \leq \dfrac{4}{5}$ on $|z| = 1$,

$$\sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{(5z)^{n+1}} = \frac{1}{5z}\sum_{n=0}^{\infty}\left[\frac{(1+z)^2}{5z}\right]^n = \frac{1}{5z}\cdot\frac{1}{1-\dfrac{(1+z)^2}{5z}} = \frac{1}{3z-1-z^2} \quad \text{uniformly on } |z| = 1,$$

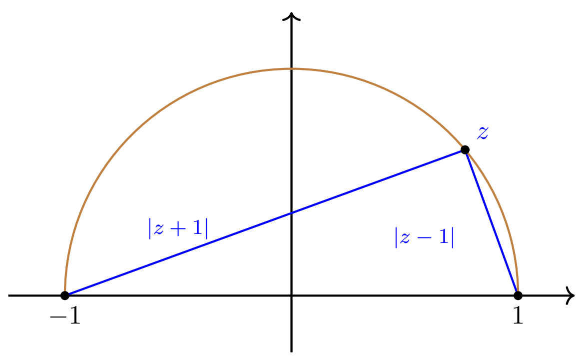
we have

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{5^n} = \frac{5}{2\pi i}\int_{|z|=1}\frac{dz}{3z-1-z^2}\,dz = 5\,\mathrm{Res}\left(\frac{1}{3z-1-z^2}; \frac{3-\sqrt{5}}{2}\right) = \sqrt{5}.$$

Example For each $k \geq 1$, since $(1+z)^n\,(1-1/z)^{2n} = \displaystyle\sum_{\ell=1}^{n}\sum_{k=1}^{2n}(-1)^k\binom{n}{\ell}\binom{2n}{k} z^{\ell}(1/z)^k$, and by the Cauchy Integral Formula, we have

$$(-1)^k\binom{n}{k}\binom{2n}{k} = \frac{1}{2\pi i}\int_C \frac{(1+z)^n\,(1-1/z)^{2n}}{z}\,dz = \frac{1}{2\pi i}\int_C \frac{[(z+1)\,(z-1)^2]^n}{z^{2n+1}}\,dz,$$

where $C = \{z \mid |z| = 1\}$.



Now

(Claim) $$\left|(z+1)\,(z-1)^2\right| \leq \frac{16}{9}\sqrt{3} \quad \text{for all } z \in C = \{z \mid |z| = 1\},$$

we have

$$\left|\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{2n}{k}\right| \leq \left(\frac{16}{9}\sqrt{3}\right)^n.$$

Proof of Claim Let $a = |z-1|$, $b = |z+1|$ and consider the function $f(a,b) = a^2 b$ for all $a \geq 0$, $b \geq 0$ such that $g(a,b) = a^2 + b^2 = 4$. By the Lagrange multiplier method, to

$$\text{maximize } f(a,b) = a^2 b \quad \text{subject to } g(a,b) = a^2 + b^2 = 4,$$

we solve

$$\begin{aligned}
&(2ab, a^2) = \nabla f = \lambda \nabla g = \lambda(2a, 2b) \\
\Longrightarrow\ & b = \lambda,\ a^2 = 2\lambda^2 \quad \text{by substituting into } g(a,b) = a^2 + b^2 = 4 \\
\Longrightarrow\ & \lambda = \frac{2}{\sqrt{3}} \text{ and } f(\sqrt{2}\lambda, \lambda) = \frac{16}{9}\sqrt{3}
\end{aligned}$$